1. Bits-1 (10 points)

Suppose we have an initial string of bits $b = b_0b_1 \cdots b_{n-1}$. There are two players, A, and B. In round $i$, A picks a set $S_i \subseteq \{0, 1, \ldots, n-1\}$. After seeing $S_i$, B chooses a number $m_i$. Now $m_i$ is added (modulo $n$) to every element of $S_i$. Call the resulting set $T_i$. Now for each element of $x \in T_i$ bit $b_x$ is flipped. This ends round $i$.

A wins if after some finite number of rounds all the bits are 0, otherwise B wins.

Show that A wins for $n$ a power of 2 and that otherwise B wins if $b / \in \{0^n, 1^n\}$.

As an example of the process suppose $n = 4$ and the initial bits of $b$ are 1000. If A picks $S_1 = \{0, 2\}$ and B picks $m_1 = 3$, then $T_1 = \{3, 1\}$. So the bit vector $b$ becomes 1101.

2. Bits-2 (10 points)

Suppose we have an initial string of bits $b = b_0b_1 \cdots b_{n-1}$. Consider the following process: if $b_{n-1} = 1$ then player A chooses a bit $x$ and replaces $b$ by $xb_0b_1 \cdots b_{n-2}$. If $b_{n-1} = 0$ then player B chooses $x$ and makes the same replacement.

Player A wins if the string eventually becomes $0^n$, otherwise B wins. Show that A always wins.

3. Hats (10 points)

There are $n$ hat wearing friends standing in a circle and black or white hats have been placed on their heads by an adversary. Initially nobody knows what color hat is on their own head. The hat wearers are allowed to think and there is a clock and after each minute passes, anybody is allowed to shout out what sort of hat they are wearing. Time is up after $n$ minutes and if no-one has declared then all will be eliminated. Furthermore, if anyone declares wrongly, the whole group will be eliminated. Can they survive with any certainty?

Now consider the same situation where someone rushes into the room and truthfully shouts “there is someone wearing a black hat” or “you are all wearing white hats”. What will happen?

Note that if $n = 10$ and five of the participants are wearing black hats, then the intruder is announcing a fact that everybody in the room already knows. How is it possible that such a pronouncement can have any effect?

[To make sure this scenario is clear, suppose there are two players B and W, where B is wearing a black hat and W is wearing a white hat. When, at time 0 an intruder yells “there is someone wearing a black hat”. One minute later B seeing that W is not wearing a black hat knows that he must be wearing a black hat, and he yells that out. What happens on the second minute?]