21-801b: Mathematical Games (CMU Fall 2022, Frieze and Sleator)

Homework 3 (due Oral in class, November 30)

Directions: Please solve these problems. Students will present the solutions in class on November 30th.

1. MIM (10 points)
The game of MIM starts with a collection of \( n \geq 2 \) piles of tokens, one of which is topped with a star. (Empty piles will be counted as piles, so the number of piles remains \( n \) throughout the game.) The two players alternate moves. If the pile with the star has zero tokens, then no move is possible. Otherwise, a move consists of removing the star and one token from the pile with the star, then the token is thrown away, and the star is moved to any other pile. The player who cannot move loses. Determine who wins, and devise an efficiently computable strategy for the winning player.

2. A Permutation Puzzle (10 points)
Consider the following puzzle, which consists of a track containing 20 numbered disks, and a turntable. The 20 disks can be pushed around the track, and the turntable can swap the four adjacent disks currently inside the turntable.

(a) [5 Points] Prove that all states can be reached.

(b) [5 Points] Suppose that instead of 20 disks there were only 19. Give an upper bound on the number of reachable states.
3. King and Rook Versus King (10 points)

KING AND ROOK VERSUS KING

Most beginning Chess players soon learn how to win this ending, so it's a surprise to find a couple of non-trivial problems which use just this material, albeit on a quarter-infinite board.

In Fig. 30, can White win? If so, in how few moves? Simon Norton says it's better to ask, "what is the smallest board (if any) that White can win on if Black is given a win if he walks off the North or East edges of the board?" Can Omar prove that it's $9 \times 11$?

Figure 31 shows Leo Moser's problem: can White win if he's allowed to make only one move with the Rook? If you find yourself frustrated by this, partition the squares in the first three columns into the four sets

- $a_1,a_3,a_5,...,c_2,c_4,c_6,...$
- $b_1,b_3,b_5,...$
- $a_2,a_4,a_6,...,c_1,c_3,c_5,...$
- $b_2,b_4,b_6,...$