1. LOL (40 points)

“LOL” is a two player game, where the two players alternate moves. The position in this game is a string of characters from the set \{ L, 0, - \}. A move consists of the replacement of a - by an L or by an 0. If a player creates a region of three consecutive characters that are “LOL” then that player immediately wins. If no moves are possible, then the game is a draw.

Clearly, starting from any position \( P \) of the game, there are only three possibilities for the outcome: The first player wins, the second player wins, or it’s a draw. Let us assume in this problem that both players play optimally, that is, the player with the winning strategy forces a win in the fewest possible moves, the player who will lose forces the game to go on as long as possible. (In case of a draw the length of the game is equal to the number of - characters in the position.) We’re interested in computing for a given position \( P \), a number \( N(P) \), which is how the number of moves the game will last.

(a) [10 Points] Suppose we start with a string of \( n \) “-” characters. Prove that if \( n \) is odd the second player cannot win, and if \( n \) is even, the first player cannot win.

(b) [10 Points] Here we extend the result in part (a). Consider an arbitrary position \( P \) which does not admit an instant winning move. Prove that if there are an odd number of “-”s then the second player cannot win, and if there are an even number of “-”s then the first player cannot win. (Hint: it may be useful to change the game slightly. Just as in chess it’s illegal to move your king into check, we can modify the game and make it forbidden to create a position from which the opponent can make an LOL. This will not change the outcome of the game.)

(c) [10 Points] Let LEFT be the player (identified in part (b)) who moves when there are an odd number of “-”s left (i.e. the player who might win). Let RIGHT be the other player.

Let’s define a game called “LOLY”, which as a slightly modified version of LOL. Both games start with the same board position. In LOLY we allow one additional kind of move. We allow LEFT to make a move that replaces L--L by LLLL. We don’t allow RIGHT to make this kind of move. We also don’t allow (as explained above) any move which would let the other player complete an LOL.

LOLY is a partizan game for which we can compute the combinatorial game value. For example, the value of - is *, the value of L--L is 1. Compute the values of the following positions:

\[
\begin{align*}
&--- \quad L--- \quad L---L \quad ----- \quad L----L \\
\end{align*}
\]

(d) [10 Points] Suppose \( P \) is this position: L-------------------L, (that’s 19 - characters) and suppose that you know that its combinatorial game value is \{\{4|3\}\{2|2\}\}1}. What is the value of \( N(P) \), and why?

Explain how this connection to combinatorial game theory might help make computing the values of \( N(P) \) more efficient.
2. The Acute Corner in Hex is a Losing Move (30 points)
Prove that (for any board size greater than 1) if the first player moves into the acute corner of the board, then the resulting position is a losing position for the first player.

3. A Hex Puzzle (30 points)
Consider the following position in a game of hex:

(a) [10 Points] Prove, by presenting an explicit strategy, that if it is White’s turn in this position, then White can win the game.

(b) [20 Points] Now suppose that it is Black’s turn in the position shown above. Prove, by presenting an explicit strategy, that Black can win the game. Here are some hints. (1) The winning move to be made by Black is B4. (2) At this point White cannot stop the black stone on A6 from connecting to the right black side of the board (this will make use of the black stone on E5). (3) After the B4 move, black has a strategy to connect the B4-C4 pair to the left black side of the board. The hard case of this argument is when White responds to B4 with D1, in which case black has a winning move at B2.

4. A Misere Hex Problem (20 points)
The following is a position in Misere Hex with White to move. Recall that White’s goal is to force Black to form a chain connecting Black’s sides, and vice versa.

Prove that White can win by playing B1. By the way, if White plays any other move then Black can force a win by playing B1.