A bound for the game chromatic number of graphs

Thomas Dinski
Department of Mathematics
Universität Bielefeld
dinski@mathematik.uni-bielefeld.de

Xuding Zhu*
Department of Applied Mathematics
National Sun Yat-sen University, Taiwan
zhu@ibm7.math.nsysu.edu.tw

Oct., 1997.†

Abstract

We show that if a graph has acyclic chromatic number k, then its game chromatic number is at most $k(k+1)$. By applying the known upper bounds for the acyclic chromatic numbers of various classes of graphs, we obtain upper bounds for the game chromatic number of these classes of graphs. In particular, since a planar graph has acyclic chromatic number at most 5, we conclude that the game chromatic number of a planar graph is at most 30, which improves the previous known upper bound for the game chromatic number of planar graphs.

*This research was supported in part by the National Science Council of the Republic of China under grant NSC87-2115-M-110-004.

†*1991 Mathematics Subject Classification.* 05C20, 05C35, 05C15

Key words and phrases. Acyclic chromatic number, game chromatic number, planar graph, partial k-tree

Appeared in Disc. Math. 196(1999) 109-115.

1 Introduction

Let G be a finite graph and let X be a set of colors. We consider a modified graph coloring problem posed as a two-person game, with one person (Alice) trying to color a graph, and the other (Bob) trying to prevent this from happening. Alice and Bob alternate turns, with Alice having the first move. A move consisting of selecting a previously uncolored vertex x and assigning it a color from the color set X distinct from the colors assigned previously (by either player) to neighbours of x. If after $n = |V(G)|$ moves, the graph G is colored, Alice is the winner. Bob wins if an impass is reached before all nodes in the graph are colored, i.e., for every uncolored vertex x and every color α from X, x is adjacent to a vertex having color α. The *game chromatic number* of a graph $G = (V, E)$, denoted by $\chi_g(G)$, is the least cardinality of a color set X for which Alice has a winning strategy. This parameter is well-defined, since Alice always wins if $|X| = |V|$.

The acyclic chromatic number of a graph is another variation of the chromatic number of a graph. Suppose $G = (V, E)$ is a graph. The acyclic chromatic number of G, denoted by $\chi_a(G)$, is the least number t so that the vertices of G can be colored by t colors in such a way that each color class is an independent set, and the subgraph of G induced by any two color classes is acyclic, i.e., the union of every two color classes induces a forest.

The acyclic chromatic number of a graph was introduced by Grünbaum [4]. It was conjectured by Grünbaum [4], and proved by Borodin [3], that the maximum acyclic chromatic number of a planar graph is equal to 5.

The game chromatic number of a graph was first studied by Bodlaeder [2]. It was proved by Faigle, Kern, Kierstead and Trotter [5] that the maximum of the game chromatic number of a forest is 4; It was proved by Kierstead and Trotter [6] that the maximum game chromatic number a planar graph is between 8 and 33. It was also proved by Kierstead and Trotter [6] that there is a function $f(n)$ such that any graph embeddable on the surface of orientable genus n has game chromatic number at most $f(n)$. However, no explicit formulae for $f(n)$ is given.

This paper explores the relation between the game chromatic number and the acyclic chromatic number of a graph. We shall prove that if a graph has acyclic chromatic number k, then its game chromatic number is at most $k(k+1)$. This implies, in particular, that the game chromatic number of a planar graph is at most 30. For $n \geq 1$, it was proved by Albertson and Berman [1] that a graph embeddable in an orientable surface of genus n has acyclic chromatic number at most $4n+4$. Therefore the game chromatic number of a graph embeddable in an orientable surface of genus n has game chromatic number at most $(4n+4)(4n+5)$. We also pose some open problems.

2 Acyclically k-colorable graphs

Theorem 1 *Let G be a graph. If $\chi_a(G) \leq k$, then $\chi_g(G) \leq k(k+1)$.*

To prove Theorem 1, we shall first describe a strategy for Alice, and then prove that it is a winning strategy.

Let c be an acyclic k-coloring of G, and let $C_1, C_2, \cdots, C_k$ be the k color classes. For $1 \leq i, j \leq k$, let $F_{i,j}$ be the forest induced by the union $C_i \cup C_j$. For $i \in \{1, 2, \cdots, k\}$, let $X_i = \{x_{i,1}, x_{i,2}, \cdots, x_{i,k+1}\}$ be a set of $k+1$ colors, and finally let $X = \cup_{i=1}^{k} X_i$ be a set of $k(k+1)$ colors. We shall describe a winning strategy for Alice in the two-person coloring game, by using the color set X to color the graph G.

In the process of the game, we say a vertex v is colored with a *correct color*, if $v \in C_i$ and the color used (by either person) is from X_i. Alice will always color a vertex with a correct color. We say a color $x_{j,q}$ is an *available color* for a vertex v (in the process of the game) if at that moment, none of the neighbours of v is colored with color $x_{j,q}$.

The strategy of Alice is as follows: For the first move, Alice arbitrarily colors a vertex with a correct color. Suppose Bob colors a vertex v, say $v \in C_i$, with a color $x_{j,q}$. If $i = j$, i.e., v is colored with a correct color, then Alice arbitrarily chooses an uncolored vertex and colors it with an available correct color. If $i \neq j$, then consider the forest $F_{i,j}$. Let T be the connected component of $F_{i,j}$ which contains v, and let S be the set of vertices of $C_i \cap T$ which have been colored by colors from X_j. If $S = \{v\}$, then Alice arbitrarily chooses an uncolored vertex and colors it with an available correct color. Otherwise, for each $u \in S - \{v\}$, let P_{vu} be the unique path of T connecting v and u. If for each $u \in S - \{v\}$, the path P_{vu} contains a colored internal vertex (i.e, a vertex which is not the end vertices of P_{vu}), then Alice arbitrarily chooses an uncolored vertex and colors it with an available correct color. Otherwise, as will be proved in Lemma 1 below, there is a unique vertex $u \in S - \{v\}$ such that every internal vertex of P_{vu} is uncolored. Then Alice chooses an internal vertex of P_{vu}, and colors it with an available correct color.

This finishes the description of the strategy of Alice. In the following we shall show that this is a winning strategy for Alice.

Lemma 1 *Let $1 \leq i, j \leq k$ be any two indices. Let T be a connected component of $F_{i,j}$ (recall that $F_{i,j}$ is the forest induced by the union $C_i \cup C_j$). At a certain step of the game, let S be the set of vertices of $T \cap C_i$ which are colored with colors from X_j. If the last move is Alice's move, then for any two vertices $u, v \in S$,* *the unique path P_{uv} of T connecting u and v contains a colored internal vertex. If the last move is Bob's move, then either for any two vertices $u, v \in S$, the path P_{uv} contains a colored internal vertex, or Bob's last move colored a vertex $v \in C_i \cap V(T)$ with a color from C_j and there is a unique vertex $u \in S$ such that every internal vertex of P_{uv} is uncolored.*

Proof. We proceed by induction on the number of turns. Initially $S = \emptyset$, and the statement is certainly true. Suppose the statement is true after the $2n$th step. (Note that the $2n$th step is Bob's move.) We need to show that it remains true after the $(2n+1)$th step, but this follows trivially from the strategy described above. Indeed, by the induction hypotheses, after the $2n$th step, there is at most one pair of vertices $u, v \in S$ such that the path P_{uv} contains no colored internal vertices. In this case, Alice colors an internal vertex of P_{uv} with an available correct color. Hence for any two vertices u, v of S, the path P_{uv} contains a colored internal vertex.

Next assume that Lemma 1 is true after the $(2n+1)$th step, we shall prove that it remains true after the $(2n+2)$th step. If in the last move, Bob did not color a vertex of $T \cap C_i$ with a color from X_j, then the conclusion of Lemma 1 certainly remains true. Assume now that in the last move Bob colored a vertex $v \in T \cap C_i$ with a color from X_j. Denote by S the S-set after the $(2n+1)$th step, and denote by S' the S-set after the $(2n+2)$th step. Then $S' = S \cup \{v\}$. For any two vertices $u, w \in S$, it follows from the induction hypotheses that the path P_{uw} contains a colored internal vertex. Assume to the contrary of Lemma 1 that there are two vertices $u, w \in S$ such that each of the paths P_{vu} and P_{vw} contains no colored internal vertex. Consider the three paths P_{uw}, P_{vu} and P_{vw}. Because T is a tree, $V(P_{uw}) \subset V(P_{vu}) \cup V(P_{vw})$. This is in contrary to the induction hypotheses that the path P_{uw} contains a colored internal vertex. This completes the proof of Lemma 1. ∎

Lemma 2 *In the process of the game, if the last step is Alice's move, then for any $1 \leq i, j \leq k$, any uncolored vertex $v \in C_j$ has at most one neighbour in C_i colored with a color from X_j; if the last move is Bob's move, then there is at most one pair of indices (i, j) such that there is exactly one uncolored vertex $v \in C_j$ which has two neighbours in C_i that are colored with colors from X_j. For every other pair (p, q) of indices, any uncolored vertex $v \in C_q$ has at most one neighbour in C_p which is colored with a color from X_q.*

Proof. We proceed by induction on the number of turns. Initially, this is trivially true. Assume that it is true after the $2n$th step, we shall prove that it remains true after the $(2n+1)$th step. The $(2n+1)$th step is Alice's move.

If after the $2n$th step, for any $1 \leq i, j \leq k$, any uncolored vertex $v \in C_j$ has at most one neighbour in C_i colored with a color from X_j, then since Alice always colors a vertex with a correct color, the lemma remains true. Assume now that after the $2n$th step, there is a unique pair of indices i, j such that there is exactly one uncolored vertex $v \in C_j$ which has two neighbours, say u, w, in C_i that are colored with colors from X_j. Then Bob's last move must have colored one of the vertices u, w, say u, with a color from X_j (by the induction hypotheses). According the strategy, Alice then considers the forest $F_{i,j}$, and the component T of $F_{i,j}$ which contains u. It follows from Lemma 1 that u, w is the unique pair of vertices in C_i such that both u, w are colored with colors from C_j and that the path P_{uw} contains no colored internal vertices. Indeed, v is the only internal vertex of P_{uw}. By the strategy, Alice colored v with an available correct color. Therefore, after Alice's move, for any $1 \leq i, j \leq k$, any uncolored vertex $v \in C_j$ has at most one neighbour in C_i colored with a color from X_j.

Finally it is obvious that if the lemma is true after the $(2n+1)$th step, then it remains true after the $(2n+2)$th step. ∎

Lemma 3 *At each step of the game, for each uncolored vertex v of G, there exists an available correct color for v.*

Proof. We shall prove it by induction on the number of turns. Initially, this is certainly true. Assume to the contrary that at a certain step, that for an uncolored vertex $v \in C_j$, there is no correct color available for v. Then the last move must be Bob's move, because Alice always color a vertex with a correct color, and hence any correct color available for an uncolored vertex before remains to be an available correct color after. Moreover, in the last move Bob must have colored a neighbour, say $u \in C_i$, of v with a color from X_j. By Lemma 2, for each $t \neq i, j$, there is at most one neighbour of v in C_t which is colored by a color from C_j. In C_i, there are at most two neighbours of v which are colored by colors from C_j. Therefore, at most k colors from C_j have been used by neigbours of v. Since $|C_j| = k + 1$, there is at least one color in C_j which is available for v, contrary to our assumption. ∎

Lemma 3 implies that the strategy for Alice described above is indeed a winning strategy. This completes the proof of Theorem 1.

Since planar graphs have acyclic chromatic number at most 5 and graphs embeddable on surfaces of orientable genus n have acyclic chromatic number at most $4n + 4$, we have the following corollary:

Corollary 1 *The game chromatic number of a planar graph is at most* 30*, and for $n \geq 1$, the game chromatic number of a graph embeddable on a surface of orientable genus n has game chromatic number at most* $(4n + 4)(4n + 5)$.

The class of k-trees is the family of graphs defined recursively as follows:

- K_k is a k-tree;
- If G is a k-tree and X is a k-clique of G, then the grpah G' obtained from G by adding a new vertex u, and connecting u to every vertex of X is a k-tree.

A graph G is a *partial k-tree* if G is a subgraph of a k-tree.

It is straightforward to verify that any k-tree, and hence every partial k-tree, has acyclic chromatic number at most $k+1$. Therefore we have the following corollary:

Corollary 2 *The game chromatic number of a partial k-tree is at most $(k+1)(k+2)$. In particular, a series-parallel graph has game chromatic number at most* 12.

3 Open Questions

The upper bound $k(k+1)$ given in Theorem 1 for the game chromatic number of graphs with acyclic chromatic number k is not tight, at least for $k=2$. A graph of acyclic chromatic number 2 is a forest. It was proved in [5] that the game chromatic number of a forest is at most 4 (and could be 4), which is less than $2(2+1)=6$. For any positive integer k, let

$$g(k) = \max\{\chi_g(G) : \chi_a(G) \leq k\}.$$

Theorem 1 implies that $g(k)$ is well-defined and $g(k) \leq k(k+1)$. It is trivial that $g(1)=1$ and the discussion above shows that $g(2)=4$. It may be a difficult problem to determine $g(k)$ for general k. It would be interesting if the value of $g(3)$ can be determined.

Question 1 *What is $g(3)$?*

A trivial observation is that $g(k+1) \geq g(k)+1$. Take any graph G whose acyclic chromatic number is k and whose game chromatic number is $g(k)$, add two vertices u and v, while u is connected to every vertices of G and v is an isolated vertex. Denote the resulting graph by G^*. Then it is obvious that G^*

has acyclic chromatic number $k+1$. We shall show that its game chromatic number is at least $g(k)+1$. Given a set $g(k)$ colors, the winning strategy for Bob (in the game of coloring the graph G^*) is the same winning strategy that Bob uses in the game of coloring G with $g(k)-1$ colors, if Alice does not color the added universal vertex u or the added isolated vertex v. In case Alice colors one of the two vertices, then Bob colors the other.

The oriented chromatic number of a graph is another variation of the coloring of graphs. Suppose $\vec{G}$ is an oriented graph. Then the oriented chromatic number $\chi_o(\vec{G})$ of $\vec{G}$ is the least number t such that the vertices of $\vec{G}$ can be colored by t colors in such a way that each color class is an independent set, and for any two color classes, say U and U', all the edges are in the same direction, i.e., either all the edges are from U to U', or all the edges are from U' to U. The oriented chromatic number $\chi_o(G)$ of an undirected graph G is the maximum of $\chi_o(\vec{G})$ among all the orientations $\vec{G}$ of G. The relation between the oriented chromatic number and the acyclic chromatic number was studied in [7, 8]. It was proved in [8] that if a graph has acyclic chromatic number k then $\chi_o(G) \leq k2^{k-1}$; and proved in [7] that if a graph G has oriented chromatic number k then $\chi_a(G) \leq k^2 + k^{3+\lceil \log_2 k \rceil + 1}$. This implies that a class of graphs has bounded acyclic chromatic number if and only if it has bounded oriented chromatic number. By simply combining Theorem 1 with the bounds given above, we have the following corollary:

Corollary 3 *If a graph G has oriented chromatic number k, then $\chi_g(G) \leq k2^{k-1}(k2^{k-1}+1)$.*

This bound seems to be much too large.

Question 2 *Find a better upper bound for the game chromatic number in terms of its oriented chromatic number.*

Analogous to the relation between the oriented chromatic number and the acyclic chromatic number, one may expect that the acyclic chromatic number of a graph could be bounded by a function of its game chromatic number. However, this is not true. For example, the complete bipartite graph $K_{n,n}$ has game chromatic number 3, but its acyclic chromatic number is $n+1$. This strange phenomenon is due to the fact that the game chromatic number of a graph is not a monotonic parameter, i.e., a subgraph H of a graph G could have a larger game chromatic number. Indeed, for any integer n, there is a graph G and a vertex v of G (resp. an edge e of G) such that the deletion of v (resp. the deletion of e) increase the game chromatic number by n. For example, let $G = (A \cup B, E)$

be the bipartite graph obtained from $K_{n,n}$ by deleting a perfect matching. Let G' be obtained from G by adding a vertex v and connect v to every vertex of B. Let G^* be obtained from G by adding one edge e connecting a vertex of A with a vertex B (in other words, G^* is obtained from $K_{n,n}$ by deleting a matching of cardinality $n-1$). Then it is straightforward to verify that $\chi_g(G') = \chi_g(G^*) = 3$, and that $\chi_g(G^* - e) = \chi_g(G' - v) = \chi_g(G) = n$.

To avoid such difficulties, we define the *hereditary game chromatic number* $\chi_{hg}(G)$ of a graph G as the maximum of the game chromatic numbers of its subgraphs, i.e.,

$$\chi_{hg}(G) = \max\{\chi_g(H) : H \text{ is a subgraphs of } G\}.$$

It follows from the definition that $\chi_{hg}(G) \geq \chi_g(G)$ for any graph G. Since the acyclic chromatic number of a graph is a monotonic parameter, Theorem 1 can be strengthened to the following:

Theorem 2 *If $\chi_a(G) \leq k$, then $\chi_{hg}(G) \leq k(k+1)$.*

Thus if a class of graphs have bounded acyclic chromatic number, then they also have bounded hereditary game chromatic number. It seems that the converse is also true, i.e., a class of graphs of bounded hereditary game chromatic number, may also have bounded acyclic chromatic number. To be precise, we have the following conjecture:

Conjecture 1 *There exists a function $\phi(n)$ such that for any graph G, if $\chi_{hg}(G) \leq n$ then $\chi_a(G) \leq \phi(n)$.*

If this conjecture is true, then for a hereditary family $\mathcal{G}$ of graphs (i.e., $G \in \mathcal{G}$ implies $H \in \mathcal{G}$ for every subgraph H of G), the following three statements would be equivalent:

1. The members of $\mathcal{G}$ have bounded acyclic chromatic number;

2. The members of $\mathcal{G}$ have bounded oriented chromatic number;

3. The members of $\mathcal{G}$ have bounded game chromatic number.

The equivalence of the first two statements were established in [7].

Note added in proof. Recently, Zhu has proved that the game chromatic number of a planar graph is at most 19 [9]; the game chromatic number of a partial k-tree is at most $3k+2$ [10]; and the game chromatic number of a graph embeddable on an orientable surface of genus n is at most $\lfloor \frac{1}{2}(3\sqrt{1+48n}+23) \rfloor$ [10]. Guan and Zhu proved [11] that the game chromatic number of an outer planar graph is at most 7.

References

[1] Albertson, M.O.;Berman, D.M.; *An Acyclic Analogue To Heawood's Theorem* Glasgow Math. J. 19 (1978), 163–166

[2] Bodlaender H. L.; *On the complexity of some coloring games*, Proceedings of the WG 1990 Workshop on Graph Theoretical Concepts in Computer Science, 1990, 30-40.

[3] Borodin,O.V.; *On acyclic colorings of planar graphs*, Discrete Math. 25 (1979), no.3,211–236

[4] Grünbaum,B.; *Acyclic colorings of planar graphs*, Israel J. Math. 14 (1973),390–408

[5] Faigle, U.; Kern, U.; Kierstead, H.; Trotter, W. T. *On the game chromatic number of some classes of graphs.* Ars Combin. 35 (1993), 143–150

[6] Kierstead, H. A.; Trotter, W. T. *Planar graph coloring with an uncooperative partner.* J. Graph Theory 18 (1994), no. 6, 569–584

[7] A. V. Kostochka, E. Sopena and X. Zhu, *Acyclic chromatic numbers of graphs*, J. Graph Theory, 24(1997), 331–340.

[8] Sopena, E. *The chromatic number of oriented graphs*, J. Graph Theory, to appear.

[9] X. Zhu, *The game coloring number of planar graphs*, J. Combin. Th. (B), to appear.

[10] X. Zhu, *Game coloring number of pseudo partial k-trees*, preprint, 1998.

[11] D. Guan and X. Zhu, *Game chromatic number of outerplanar graphs*, J. Graph Theory, to appear.