The Problem of Coincidences

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Summary

This paper deals with the origin, history and various appearances of the problem of coincidences (matches, rencontres) in the theory of probability.

1. Introduction

The problem of coincidences (matches, rencontres) originates in the game of thirteen (jeu du treize) and was proposed in 1708 by PIERRE RÉMOND DE MONTMORT (1678–1719) in his book [38 p. 185]. In a letter to JOHANN BERNOULLI (1667–1748), dated November 15, 1710, MONTMORT gave the solution of the problem which he proposed in 1708. He added that he had found the general solution of the game of thirteen, but it would be too long to give all the details. Actually, he did not even write down the general solution. The general solution was given by NIKOLAUS BERNOULLI (1687–1759), a nephew of JOHANN BERNOULLI, in a letter to MONTMORT, dated February 26, 1711. Both letters are reprinted in the second edition of MONTMORT's book [38]. See [39] pp. 303–307 and pp. 308–314 respectively. MONTMORT's problem had great influence on the development of probability theory, and the aim of this paper is to give a historical account of the result of this influence.

2. The Game of Thirteen

The game of thirteen (jeu du treize) is played by any number of players with a French deck of 52 cards which consists of an ace (1), nine numerals (2 to 10), knave (11), queen (12) and king (13) in each of four suits (spades, hearts, diamonds, and clubs). The rank of each card is indicated in brackets. First, the players choose a banker. The banker shuffles the cards and turns up thirteen cards one after the other. As he turns up the cards he calls out the names of the cards in a suit in order of rank (ace, two, ..., king). If in this sequence no card coincides with the name of the card called, the banker pays each one of the players and yields the bank to the player on his right. But if there is a coincidence in the turning of the thirteen cards, for example, if an ace turns up at the first step when the banker has called ace, or a two at the second step when the banker has called two, the banker takes all the stakes and begins again as before.

In 1708 at the end of his book MONTMORT [38, pp. 185–189], [39, p. 278] proposed four problems for solution. The first problem was to determine the probability that the banker wins a game in the game of thirteen.

More generally, it can be asked what is the probability that the banker wins a game if instead of 13 there are n cards in a suit and if instead of 4 there are s suits in the deck, and in each game the banker names n cards in a suit.

It is convenient to study the two cases, $s=1$ and $s\geqq 1$, separately and to use the following models:

Model I. *A box contains n cards marked* $1,2,\ldots,n$, *and all the n cards are drawn from the box one by one without replacement. Every outcome of this random trial can be represented by a permutation of* $1,2,\ldots,n$. *It is assumed that all the* $n!$ *permutations of* $1,2,\ldots,n$ *are equally probable.*

We say that a coincidence occurs at the i^{th} drawing if the card drawn is marked i ($i=1,2,\ldots,n$). Denote by $P(n,k)$ the probability that in the n drawings we have at least k coincidences ($k=0,1,\ldots,n$). We write

$$P(n,1)=D(n)/n!, \tag{1}$$

where $D(n)$ denotes the number of permutations of $1,2,\ldots,n$ in which at least one coincidence occurs, and

$$1-P(n,1)=Q(n)/n!, \tag{2}$$

where $Q(n)$ denotes the number of permutations of $1,2,\ldots,n$ in which no coincidence occurs.

Model II. *A box contains s sets of cards, each set consisting of n cards marked* $1,2,\ldots,n$. *From the box n cards are drawn one by one without replacement. There are* $ns(ns-1)\ldots(ns-n+1)$ *possible outcomes and these are assumed to be equally probable.*

We say again that a coincidence occurs at the i^{th} drawing if the card drawn is marked i ($i=1,2,\ldots,n$). Denote by $P(n,s,k)$ the probability that in the n drawings we have at least k coincidences ($k=0,1,\ldots,n$). Of course $P(n,1,k)=P(n,k)$ for $0\leqq k\leqq n$.

By using the above model we can interpret $P(13,4,1)$ as the probability that the banker wins a game in the game of thirteen. Thus the solution of MONTMORT's problem consists in the determination of $P(13,4,1)$.

In the first edition of his book MONTMORT [38 pp. 54–64] discussed in detail a simplified version of the game of thirteen, namely, a game which is played with only one suit of 13 cards. The probability that the banker wins a game is $P(13,1,1)=P(13,1)$.

In what follows we shall summarize first the basic results for $P(n,k)$ and $P(n,s,k)$ and then we shall discuss their historical development.

3. Basic Results for Model I

Define A_i $(i=1,2,\dots,n)$ as the event that a coincidence occurs at the i^{th} drawing. Denote by ξ_n the number of coincidences in the n drawings and let $\xi_0 = 0$.

We shall give two methods for finding the distribution of ξ_n for $n=1,2,\dots$. One is based on a recurrence formula, and the other on a general theorem of probability.

Obviously

$$P\{\xi_n = k\} = \binom{n}{k} Q(n-k)/n! \tag{3}$$

where $Q(n)$ is defined by (2) for $n \geqq 1$ and $Q(0)=1$. Thus the problem of finding the distribution of ξ_n can be reduced to the problem of finding $Q(j)$ for $j = 0,1,\dots,n$. Since

$$\sum_{k=0}^{n} P\{\xi_n = k\} = 1, \tag{4}$$

it follows from (3) that

$$\sum_{j=0}^{n} \frac{Q(n-j)}{j!(n-j)!} = 1 \tag{5}$$

for $n=0,1,2,\dots$. Formula (5) is already a recurrence formula for the determination of $Q(n)$ $(n=0,1,2,\dots)$. However, it is easy to find an explicit expression for $Q(n)$. Multiplying (5) by x^n and summing for $n=0,1,2,\dots$, we get

$$\sum_{n=0}^{\infty} \frac{Q(n)}{n!} x^n = e^{-x}(1-x)^{-1} \tag{6}$$

for $|x|<1$. Hence

$$\frac{Q(n)}{n!} = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \tag{7}$$

for $n=0,1,2,\dots$. The same result can also be obtained by using the recurrence formula

$$Q(n) = (n-1)\left[Q(n-1) + Q(n-2)\right] \tag{8}$$

for $n \geqq 2$ where $Q(0)=1$ and $Q(1)=0$.

By a general theorem of C. Jordan [24] if $A_1, A_2, \dots, A_n$ are random events and ξ_n denotes the number of events occurring among $A_1, A_2, \dots, A_n$, then

(9) $$P\{\xi_n=k\}=\sum_{r=k}^{n}(-1)^{r-k}\binom{r}{k}B_r$$

for $k=0,1,\dots,n$ and

(10) $$P\{\xi_n\geqq k\}=\sum_{r=k}^{n}(-1)^{r-k}\binom{r-1}{k-1}B_r$$

for $k=1,2,\dots,n$ where $B_0=1$ and

(11) $$B_r=\sum_{1\leqq i_1<i_2<\dots<i_r\leqq n}P\{A_{i_1}A_{i_2}\dots A_{i_r}\}$$

for $r=1,2,\dots,n$. We can interpret B_r as the r^{th} binomial moment of ξ_n, that is,

(12) $$B_r=E\left\{\binom{\xi_n}{r}\right\}$$

for $r=0,1,\dots,n$. See also TAKÁCS [49].

For Model I

(13) $$P\{A_{i_1}A_{i_2}\dots A_{i_r}\}=\frac{(n-r)!}{n!}$$

and

(14) $$B_r=1/r!$$

if $r=1,2,\dots,n$. Thus by (10)

(15) $$P(n,k)=P\{\xi_n\geqq k\}=\frac{1}{(k-1)!}\sum_{r=k}^{n}\frac{(-1)^{r-k}}{r(r-k)!}$$

for $k=1,2,\dots,n$.

Since

(16) $$\lim_{n\to\infty}\frac{Q(n)}{n!}=e^{-1},$$

it follows from (3) that

(17) $$\lim_{n\to\infty}P\{\xi_n=k\}=e^{-1}/k!$$

for $k=0,1,2,\dots$.

4. Basic Results for Model II

Define A_i $(i=1,2,\dots,n)$ again as the event that a coincidence occurs at the i^{th} drawing. Denote by $\xi_n(s)$ the number of coincidences in the n drawings and let $\xi_0(s)=0$.

In this case

$$\boldsymbol{P}\{A_{i_1} A_{i_2} \dots A_{i_r}\} = \frac{s^r}{ns(ns-1)\dots(ns-r+1)} \tag{18}$$

for $r=1,2,\dots,n$ and by (9) we obtain that

$$\boldsymbol{P}\{\xi_n(s)=k\} = \frac{1}{k!} \sum_{r=k}^{n} \frac{(-1)^{r-k}}{(r-k)!} \frac{ns(ns-s)\dots(ns-rs+s)}{ns(ns-1)\dots(ns-r+1)} \tag{19}$$

for $k=0,1,\dots,n$ and by (10)

$$\boldsymbol{P}\{\xi_n(s) \geqq k\} = \frac{1}{(k-1)!} \sum_{r=k}^{n} \frac{(-1)^{r-k}}{r(r-k)!} \frac{ns(ns-s)\dots(ns-rs+s)}{ns(ns-1)\dots(ns-r+1)} \tag{20}$$

for $k=1,2,\dots,n$. By (19) we obtain that

$$\lim_{n\to\infty} \boldsymbol{P}\{\xi_n(s)=k\} = e^{-1}/k! \tag{21}$$

for $k=0,1,2,\dots$ and for any $s=1,2,\dots$.

5. Montmort's Results

First, in 1708 MONTMORT [38, pp. 54–64] discussed the problem of finding the probability

$$P(n, 1 = D(n)/n! \tag{22}$$

for Model I and for $n=13$. For brevity, we shall write

$$P^*(n) = D(n)/n! \tag{23}$$

that is, $P^*(n)$ is the probability that at least one coincidence occurs in the trial described in Model I.

Without giving any justification MONTMORT stated that

$$P^*(n) = \frac{(n-1)\,P^*(n-1) + P^*(n-2)}{n} \tag{24}$$

for $n=2,3,\dots$ where $P^*(0)=0$ and $P^*(1)=1$. By using the recurrence formula (24) MONTMORT calculated $P^*(n)$ for $n \leqq 13$ and obtained that

$$P^*(13) = \frac{109339663}{172972800}. \tag{25}$$

Furthermore, MONTMORT stated that

$$P^*(n) = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i!} \tag{26}$$

for $n \geqq 1$, and concluded that

$$\lim_{n \to \infty} P^*(n) = 1 - e^{-1}. \tag{27}$$

See also MONTMORT [39, pp. 130–143].

MONTMORT's result (24) aroused the interest of several mathematicians and now we have various proofs for (24). Formula (26) follows easily from (24). If we write (24) in the form of

$$P^*(n) - P^*(n-1) = -[P^*(n-1) - P^*(n-2)]/n \tag{28}$$

where $n \geqq 2$ and apply (28) repeatedly, then we obtain

$$P^*(n) - P^*(n-1) = (-1)^{n-1}/n! \tag{29}$$

for $n \geqq 1$ and this implies (26). The limit (27) is an immediate consequence of (26).

One wonders how MONTMORT arrived at (24). It seems likely that he obtained (24) empirically.

The first edition of MONTMORT's book [38] is discussed briefly by F.N. DAVID [8, pp. 140–160], and the second edition [39] by I. TODHUNTER [50, pp. 78–134]. See, in particular, pp. 91–93, 105, 115, 116, 120–121, 122–125 in [50]. See also K. JORDAN [25, pp. 431–441], [26, pp. 429–439].

6. Johann Bernoulli's Contribution

MONTMORT sent a copy of his book [38] to JOHANN BERNOULLI for perusal. In his letter to MONTMORT, dated March 17, 1710, BERNOULLI commented on the book and his letter is reprinted in the second edition of MONTMORT's book [39, pp. 283–298] published in 1713.

JOHANN BERNOULLI believed that the solution of MONTMORT's problem was impossible because the lengthy calculations needed to find $P(13, 4, 1)$ could not be finished in a lifetime [39, p. 298]. He remarked [39, p. 290] that MONTMORT's formula (26), which is beautiful and curious, can easily be obtained from (24). Indeed if we write (24) in the form of (28), then we get (26) easily.

In his letter, JOHANN BERNOULLI [39, p. 290] mentioned also that

$$P^*(n) = \sum_{j=0}^{n} \frac{1}{j!} - \sum_{j=0}^{n} \frac{P^*(n-j)}{j!} \tag{30}$$

for $n \geqq 1$. This equation is trivially true. If we put (3) into (4), then we get (30). It seems that BERNOULLI did not notice that (30) is obviously true, and that it implies MONTMORT's formula (26).

In his reply to BERNOULLI, dated November 15, 1710, MONTMORT [39, pp. 303–307] wrote that he had found the general solution of the game of thirteen and that by his calculations [39, p. 304]

(31) $$2P(13,4,1)-1=\frac{69056823706869897}{241347817621535625}.$$

Actually, MONTMORT made a slight error in calculating (31). For a correction see N. BERNOULLI [39, p. 324].

7. Nikolaus Bernoulli's Contribution

Along with JOHANN BERNOULLI's comments, his nephew. NIKOLAUS BERNOULLI also sent his remarks to MONTMORT. These remarks are reprinted in the second edition MONTMORT's book [39, pp. 299–303]. N. BERNOULLI gave two proofs for MONTMORT's formula (26). Both proofs are remarkable and demonstrate the ingenuity of their inventor.

N. BERNOULLI [39, p. 301] gave the following proof for (26). In (26) we can write

(32) $$D(n)=\sum_{j=1}^{n} D(n,j)$$

where $D(n,j)$ is the number of permutations of $1,2,\ldots,n$ in which there is a coincidence at the j^{th} place and there is no coincidence before. He demonstrated that

(33) $$D(n,j)=\sum_{i=0}^{j-1}(-1)^i\binom{j-1}{i}(n-1-i)!$$

for $j=1,2,\ldots,n$. From (33) he concluded that

(34) $$\sum_{j=1}^{m} D(n,j)=\sum_{i=0}^{m-1}(-1)^i\binom{m}{i+1}(n-1-i)!$$

for $m=1,2,\ldots,n$. If $m=n$ in (34), we get $D(n)$ and (26) follows from (23).

Formula (33) can also be proved by using the recurrence formula

(35) $$D(n,j+1)=D(n,j)-D(n-1,j)$$

for $n=2,3,\ldots$ and $j=1,2,\ldots,n-1$ where $D(n,1)=(n-1)!$ for $n=1,2,\ldots$. To obtain (35), we divide the set of permutations of $1,2,\ldots,n$ in which there is no coincidence at the $2^{\text{nd}},3^{\text{rd}},\ldots,j^{\text{th}}$ places and there is a coincidence at the $(j+1)^{\text{st}}$ place, into two disjoint subsets such that in the first set the first element is 1 and in the second set the first element is $\neq 1$.

In his second proof of (26) N. BERNOULLI [39, pp. 301–302] demonstrated that

(36) $$P^*(n)=\frac{1}{n}+\frac{(n-1)}{n}\frac{[D(n-1)-(n-2)!+D(n-2)]}{(n-1)!}$$

which is the same as (24). The left-hand side of (36) is the probability that at least one coincidence occurs in a permutation of $1,2,\ldots,n$ chosen at random.

This event can occur in two mutually exclusive ways: The first number chosen is 1 which has probability $1/n$, or the first number is different from 1, which has probability $(n-1)/n$, and at least one coincidence occurs among the remaining $n-1$ elements. The proof of (36) is complete if we prove that the number of permutations of $1,2,\ldots,n$ in which the first element is a fixed integer $i \neq 1$ and at least one coincidence occurs among the remaining $n-1$ elements is $D(n-1)-(n-2)!+D(n-2)$. There are two possibilities: The i^{th} element of a permutation is 1 or $\neq 1$. In the first case, the number of permutations in which at least one coincidence occurs is $D(n-2)$. For in each permutation the first element is $i \neq 1$ and the i^{th} element is 1 and the remaining $(n-2)$ elements can be arranged in $D(n-2)$ ways such that at least one coincidence occurs. In the second case, the number of permutations in which at least one coincidence occurs is $D(n-1)-(n-2)!$. For in each permutation the first element is $i \neq 1$ and the i^{th} element is, say, $j \neq 1$, and the remaining $(n-2)$ elements can be arranged in $D(n-1)-(n-2)!$ ways such that at least one coincidence occurs among them. To see this let us interchange the first and the i^{th} element in each permutation and remove the i^{th} element which is i. Since $j \neq 1$, the new first element cannot be 1. Thus in each permutation we have $n-1$ elements and from the $D(n-1)$ permutations in which at least one coincidence occurs we should remove $(n-2)!$ permutations in which the first element is 1. This completes the proof of (36) and (24). Finally, N. BERNOULLI indicated how (26) can be deduced from (24).

NIKOLAUS BERNOULLI's masterly use of conditional probabilities in the proof of (36) is truly remarkable.

The general solution of the game of thirteen, $P(13,4,1)$, can be obtained as a straightforward extension of N. BERNOULLI's formula (34). It would be interesting to know whether MONTMORT obtained (31) before or after reading BERNOULLI's remarks. In any case, in a letter to MONTMORT, dated February 26, 1711, N. BERNOULLI [39, pp. 308–314] gave also the solution of the general problem (pp. 308–309). Unfortunately, BERNOULLI's formula contained an oversight which MONTMORT [39, pp. 315–323] essentially corrected in his answer to N. BERNOULLI, dated April 10, 1711. (See [39, p. 315].) Afterwards, BERNOULLI [39, pp. 323–337] in his letter to MONTMORT, dated November 10, 1711, gave the correct formula,

$$(37) \qquad P(n,s,1)=\sum_{r=1}^{n}(-1)^{r-1}\binom{n}{r}\frac{s^r}{ns(ns-1)\ldots(ns-r+1)},$$

for the probability of having at least one coincidence in the general case ([39, pp. 323–324]). If $n=13$ and $s=4$ in (37), we get MONTMORT's result (31). Formula (37) is a particular case of (20).

8. The Method of Inclusion and Exclusion

The proof of MONTMORT's formula (26), which N. BERNOULLI [39, p. 301] sent to MONTMORT on March 17, 1710 is very significant in the theory of probability.

By N. BERNOULLI's results we can conclude that if n events $A_1, A_2, \dots, A_n$ satisfy the symmetry properties

$$\text{(38)} \qquad \boldsymbol{P}\{A_{i_1} A_{i_2} \dots A_{i_r}\} = \boldsymbol{P}\{A_1 A_2 \dots A_r\}$$

for $1 \leqq i_1 < i_2 < \dots < i_r \leqq n$, then

$$\text{(39)} \qquad \boldsymbol{P}\{A_1 + A_2 + \dots + A_n\} = \sum_{r=1}^{n} (-1)^{r-1} \binom{n}{r} \boldsymbol{P}\{A_1 A_2 \dots A_r\}.$$

By following BERNOULLI's proof, we can write

$$\text{(40)} \quad \boldsymbol{P}\{A_1 + A_2 + \dots + A_n\} = \boldsymbol{P}\{A_1\} + \boldsymbol{P}\{\bar{A}_1 A_2\} + \dots + \boldsymbol{P}\{\bar{A}_1 \bar{A}_2 \dots \bar{A}_{n-1} A_n\}.$$

For among $A_1, A_2, \dots, A_n$ at least one event can occur in several mutually exclusive ways: The first event occurring in the sequence $A_1, A_2, \dots, A_n$ is A_i where $i = 1, 2, \dots, n$. In (40)

$$\text{(41)} \qquad \boldsymbol{P}\{\bar{A}_1 A_2\} = \boldsymbol{P}\{A_2\} - \boldsymbol{P}\{A_1 A_2\}.$$

If we apply formula (41) repeatedly in evaluating the j^{th} term on the right-hand side of (40), then we get

$$\text{(42)} \qquad \boldsymbol{P}\{\bar{A}_1 \dots \bar{A}_{j-1} A_j\} = \sum_{r=0}^{j-1} (-1)^r \binom{j-1}{r} \boldsymbol{P}\{A_1 A_2 \dots A_r\}$$

for $j = 1, 2, \dots, n$. This formula can also be proved by mathematical induction. By (40) and (42) we get (39).

The above method works also in the case where (38) does not hold. In this case the right-hand side of (39) is given by (10) with $k = 1$.

If A_i $(i = 1, 2, \dots, n)$ denotes the event that a coincidence occurs at the i^{th} drawing in Model II, then

$$\text{(43)} \qquad \boldsymbol{P}\{A_{i_1} A_{i_2} \dots A_{i_r}\} = s^r / (ns(ns-1) \dots (ns-r+1))$$

for $r = 1, 2, \dots, n$ and by (39) we get N. BERNOULLI's formula (37). If, in particular, $s = 1$, we get MONTMORT's formula (26).

The problem of coincidences was also discussed by A. DE MOIVRE [9, pp. 59–66], [10, pp. 95–103], [11, pp. 109–117] in his book on probability theory, the first edition of which was published in 1718. DE MOIVRE demonstrated that for Model II the probability $P(n, s, k)$ is given by (20) where $k = 1, 2, \dots, n$. If $k = 1$, then (20) reduces to N. BERNOULLI's formula (37).

DE MOIVRE's result (20) can be formulated in the following more general way. Let $A_1, A_2, \dots, A_n$ be n random events which satisfy the symmetry property (38) for $r = 1, 2, \dots, n$. Denote by ξ_n the number of events occurring among $A_1, A_2, \dots, A_n$. Then

$$\text{(44)} \qquad \boldsymbol{P}\{\xi_n \geqq k\} = \binom{n}{k} \sum_{r=k}^{n} (-1)^{r-k} \binom{n-k}{r-k} \frac{k}{r} \boldsymbol{P}\{A_1 A_2 \dots A_r\}$$

for $k=1,2,\ldots,n$. DE MOIVRE's proof is based on the following formula:

$$\mathbf{P}\{A_1 \ldots A_k \bar{A}_{k+1} \ldots \bar{A}_n\} = \sum_{r=k}^{n} (-1)^{r-k} \binom{n-k}{r-k} \mathbf{P}\{A_1 A_2 \ldots A_r\} \tag{45}$$

which can be proved in the same way as (42). Actually, (45) is an immediate consequence of (42). Since obviously

$$\mathbf{P}\{\xi_n = k\} = \binom{n}{k} \mathbf{P}\{A_1 \ldots A_k \bar{A}_{k+1} \ldots \bar{A}_n\} \tag{46}$$

for $k=1,2,\ldots,n$, we get (44) by summing (46) for $k, k+1, \ldots, n$.

The above method of finding the distribution of ξ_n works also in the case where (38) does not hold. In this case the right-hand side of (44) is given by (10).

Formulas (9) and (10) were found in 1867 by C. JORDAN [24]. It was indeed a great achievement by N. BERNOULLI and DE MOIVRE to prove (9) and (10) in an important particular case in 1710 and 1718 respectively.

9. Euler's Contribution

L. EULER wrote two papers on the problem of coincidences. In his paper of 1751 [13] he gave a proof for MONTMORT's formula (26) by using the same approach as N. BERNOULLI used in 1710. EULER obtained also the limit (27). In his paper of 1779 L. EULER [14] proved that if $Q(n)$ denotes the number of permutations of $1,2,\ldots,n$ in which no coincidence occurs, then

$$Q(n) = (n-1)\,[Q(n-1) + Q(n-2)] \tag{47}$$

for $n \geqq 2$ where $Q(0)=1$ and $Q(1)=0$. From (47) he derived the equation

$$Q(n) - Q(n-1) = \frac{(-1)^n}{n} \tag{48}$$

for $n=1,2,\ldots$. Since (47) is equivalent to MONTMORT's equation (24) and since (48) implies (26), EULER's results prove MONTMORT's formulas (24) and (26).

EULER's proof of (47) is along the same lines as N. BERNOULLI's proof of (24) in 1710. In the set of permutations of $1,2,\ldots,n$ in which no coincidence occurs the first element can only be $i=2,3,\ldots,n$. The set in which the first element is $i\,(i \neq 1)$ can be divided into two disjoint subsets. In the first subset the i^{th} element is 1, and in the second subset, the i^{th} element is j where $j \neq 1$. The number of permutations in the first subset is obviously $Q(n-2)$. For if we remove the first element (i) and the i^{th} element (1) from each permutation, the remaining $n-2$ elements can be arranged in $Q(n-2)$ ways such that no coincidence occurs. The number of permutations in the second subset is $Q(n-1)$. To see this let us interchange the first element (i) and the i^{th} element (j) in each permutation, and remove the new i^{th} element (i). The number of permutations obtained in this way is $Q(n-1)$. Since i may take the $n-1$ values $2,3,\ldots,n$, we get (47). See also E. NETTO [42, pp. 66–67].

In proving (24) N. BERNOULLI [39, pp. 301–302] essentially demonstrated that

$$D(n)=(n-1)[D(n-1)+D(n-2)] \tag{49}$$

for $n \geqq 2$ where $D(0)=0$ and $D(1)=1$. Equations (47) and (49) are equivalent; however, (47) can be proved in a shorter way than (49).

10. Catalan's Contribution

MONTMORT's formula (3) was also proved by J. H. LAMBERT [33] in 1771. In 1812 P.S. LAPLACE [34, pp. 217–225], [35, pp. 219–228] derived N. BERNOULLI's formula (37) and gave an approximating expression for (37). In 1837 E. CATALAN [6] studied Model I. He learned of the results of EULER [13] and LAPLACE [34, pp. 217–225] only after his paper was finished. He showed that in formula (3), $Q(n)$ satisfies equation (8), and from (8) he derived formula (7) thus proving MONTMORT's result (26) again. CATALAN's proof is only a slight variant of N. BERNOULLI's proof or EULER's.

CATALAN considered also a generalization of Model I by assuming that instead of n we take out m cards from the box where $1 \leqq m \leqq n$. If $m=n$, we obtain Model I. Denote by $\eta_m(n)$ the number of coincidences in the m drawings. CATALAN proved that if all the $n(n-1)\dots(n-m+1)$ outcomes are equally probable, then

$$\boldsymbol{P}\{\eta_m(n)=k\} = \binom{m}{k}(n-m)!\,Q(n-k,m-k)/n! \tag{50}$$

for $k=0,1,2,\dots,m$ where $Q(n,m)$ is the number of permutations of $1,2,\dots,n$ in which no coincidence occurs in the first m places. He showed that if $1 \leqq m \leqq n$, then

$$Q(n,m)=(n-m+1)\,Q(n,m-1)-Q(n-1,m-1) \tag{51}$$

where $Q(n,0)=1$ for $n \geqq 0$. By solving this difference equation CATALAN proved that

$$Q(n,m)=\frac{1}{(n-m)!}\sum_{r=0}^{m}(-1)^r\binom{m}{r}(n-r)! \tag{52}$$

for $0 \leqq m \leqq n$. The same result can also be obtained by making use of formula (44).

11. The Number of Non-Zero Terms in a Determinant

Let $A_n=[a_{ij}]_{i,j}$ be an $n \times n$ matrix. The determinant of A_n is defined as

$$\operatorname{Det}(A_n)=\sum_{(i_1,i_2,\dots,i_n)\in P_n}(-1)^{I(i_1,i_2,\dots,i_n)}\,a_{1i_1}a_{2i_2}\dots a_{ni_n} \tag{53}$$

where P_n is the set of all the $n!$ permutations of $1, 2, \ldots, n$ and $I(i_1, i_2, \ldots, i_n)$ is the number of inversions in the permutation $(i_1, i_2, \ldots, i_n)$. By following TH. MUIR [40, p. 19] we define the permanent of A_n by

$$\text{Per}(A_n) = \sum_{(i_1, i_2, \ldots, i_n) \in P_n} a_{1i_1} a_{2i_2} \ldots a_{ni_n}. \tag{54}$$

In 1872 J.J. WEYRAUCH [53] showed that in (53) and in (54) there are $\binom{n}{k} Q(n-k)$ terms which contain exactly k diagonal elements $(k = 0, 1, \ldots, n)$ where $Q(n)$ is the number of permutations of $1, 2, \ldots, n$ in which no coincidence occurs. $Q(n)$ is given by (7) for $n \geqq 0$. See also R. BALTZER [1]. WEYRAUCH's result can also be formulated in the following way: If $A_n(x)$ is an $n \times n$ matrix whose diagonal elements are x and whose off-diagonal elements are 1, then

$$\text{Per}(A_n(x)) = \sum_{k=0}^{n} \binom{n}{k} Q(n-k) x^k. \tag{55}$$

In particular, we have

$$\text{Per}(A_n(0)) = Q(n). \tag{56}$$

In 1891 G. DE LONGCHAMPS [36] proved that if A_n is an $n \times n$ matrix in which m diagonal elements are 0 and every other element is different from zero, then (53) and (54) contain precisely $Q(n, m)(n-m)!$ non-zero terms where $Q(n, m)$ is given by (52).

DE LONGCHAMPS [36] proposed the problem of finding a formula for the number of nonzero terms in (53) or (54) for a matrix A_n in which some selected elements are equal to 0. This problem was solved in 1891 by C.A. LAISANT [32]; see also A. HOLTZE [22], E. NETTO [42, pp. 71–74], [43, pp. 71–74], E.G. OLDS [44], J. RIORDAN [46, p. 184], and J. RYSER [47, pp. 22–28].

Formulas (55) and (56) are trivially true. If we use MONTMORT's formula (7) for $Q(n)$ $(n \geqq 0)$, then (55) and (56) can be expressed in explicit forms. Conversely, we can use (55) or (56) for the determination of $Q(n)$ for $n \geqq 0$. In 1951 J.W. BOWER [4], and in 1973 J.J. JOHNSON [23] proved MONTMORT's formula (7) in such a way. In 1946 I also determined $Q(n)$ by using (56). I attended the classes of Professor K. JORDAN on probability theory where I learned about MONTMORT's results (24) and (26) which were stated without proof in MONTMORT's book [38]. If $Q(n)$ is defined by (2), then by MONTMORT's results we have

$$Q(n) = (n-1)[Q(n-1) + Q(n-2)] \tag{57}$$

for $n \geqq 2$ where $Q(0) = 1$ and $Q(1) = 0$, and conversely (57) implies (24) and (26). It occurred to me immediately that $Q(n)$ can be interpreted as the number of terms which contain no diagonal elements in a determinant of order n, that is, $Q(n)$ is the permanent of an $n \times n$ matrix whose diagonal elements are 0 and whose off-diagonal elements are 1. By using this interpretation of $Q(n)$ I gave the following proof for (57): Denote by $Q^*(n)$ the permanent of an $n \times n$ matrix whose elements are 1 except the $2^{nd}, 3^{rd}, \ldots, n^{th}$ diagonal elements which are 0. By expanding both permanents according to the first column we obtain

$$Q(n) = (n-1)\, Q^*(n-1) \tag{58}$$

and

$$Q^*(n) = Q(n-1) + (n-1)\, Q^*(n-1) = Q(n-1) + Q(n) \tag{59}$$

for $n \geqq 1$. If we express $Q^*(n-1)$ in (58) by (59), we get (57) which was to be proved. I communicated my results to Professor JORDAN. In his letter, dated April 28, 1946, Professor JORDAN wrote me that he intended to refer to these results in his forthcoming book on probability theory which he had recently finished; see K. JORDAN [25, p. 437], [26, p. 435].

12. Some Generalizations

If we assume that in Model II the number of 1's, 2's, ..., n's are not necessarily the same, then we arrive at a generalization of the problems discussed in this paper. Such general models have been studied by E.G. OLDS [44], J.A. GREENWOOD [19], I. KAPLANSKY [28], [29], I.L. BATTIN [3], I. KAPLANSKY & J. RIORDAN [30], and others.

There is a large number of papers on the statistical applications of coincidence problems. Here I mention only a paper by TH. YOUNG [55] which was published as early as 1819. See an appraisal of this paper by M.G. KENDALL [31].

The problem of coincidences, its generalizations and its various applications are discussed in several books and in many papers. I mention the books of W.A. WHITWORTH [54, pp, 100–117], E. LUCAS [37, pp. 211–215, 490–491], E. NETTO [42, pp. 66–74], [43, pp. 66–74], M. FRÉCHET [17, pp. 135–148], J. RIORDAN [46, pp. 57–62], H.J. RYSER [47, pp. 22–28] and K. JORDAN [25, pp. 431–441], [26, pp. 429–439]. In 1943 M. FRÉCHET [18] published a detailed study on coincidences. See also M. CANTOR [5], A. CAYLEY [7], S. KANTOR [27], P. SEELHOFF [48], G. MUSSO [41], E. BATICLE [2], E. ESCARDÓ [12], M. FRÉCHET [15], J. HAAG [20], I.B. HAÁZ [21], M. ORTS [45], J. TOUCHARD [51], [52] and others.

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