Subtraction Games

$S \subseteq \{1, 2, 3, \dots \}$

Position n.

Move: choose $x \in S$

$$n \longrightarrow n - x$$

Case 1

$|S| < \infty$

Let $g(n)$ be the Grundy number of n.

g is ultimately periodic if

$\exists\, n_0, s$ such that

$n \geq n_0$ implies $g(n+s) = g(n)$.

Thm

$|S| < \infty \Rightarrow g$ is ultimately periodic

Proof

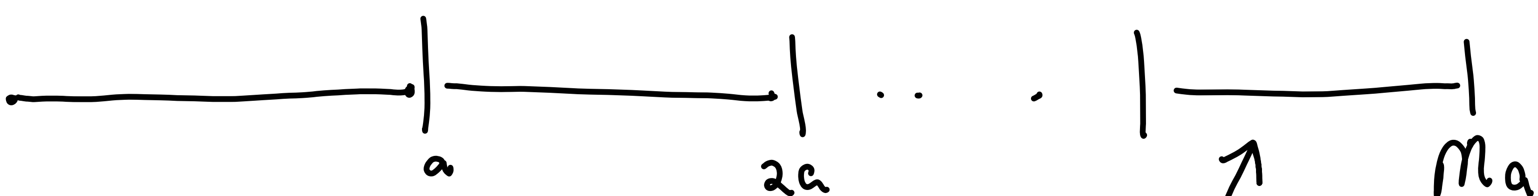
Let $S = \{a_1 < a_2 < \cdots < a_k = a\}$

(i) $\quad \operatorname{mex}(T) \leq |T| \quad$ for all $|T| < \infty$

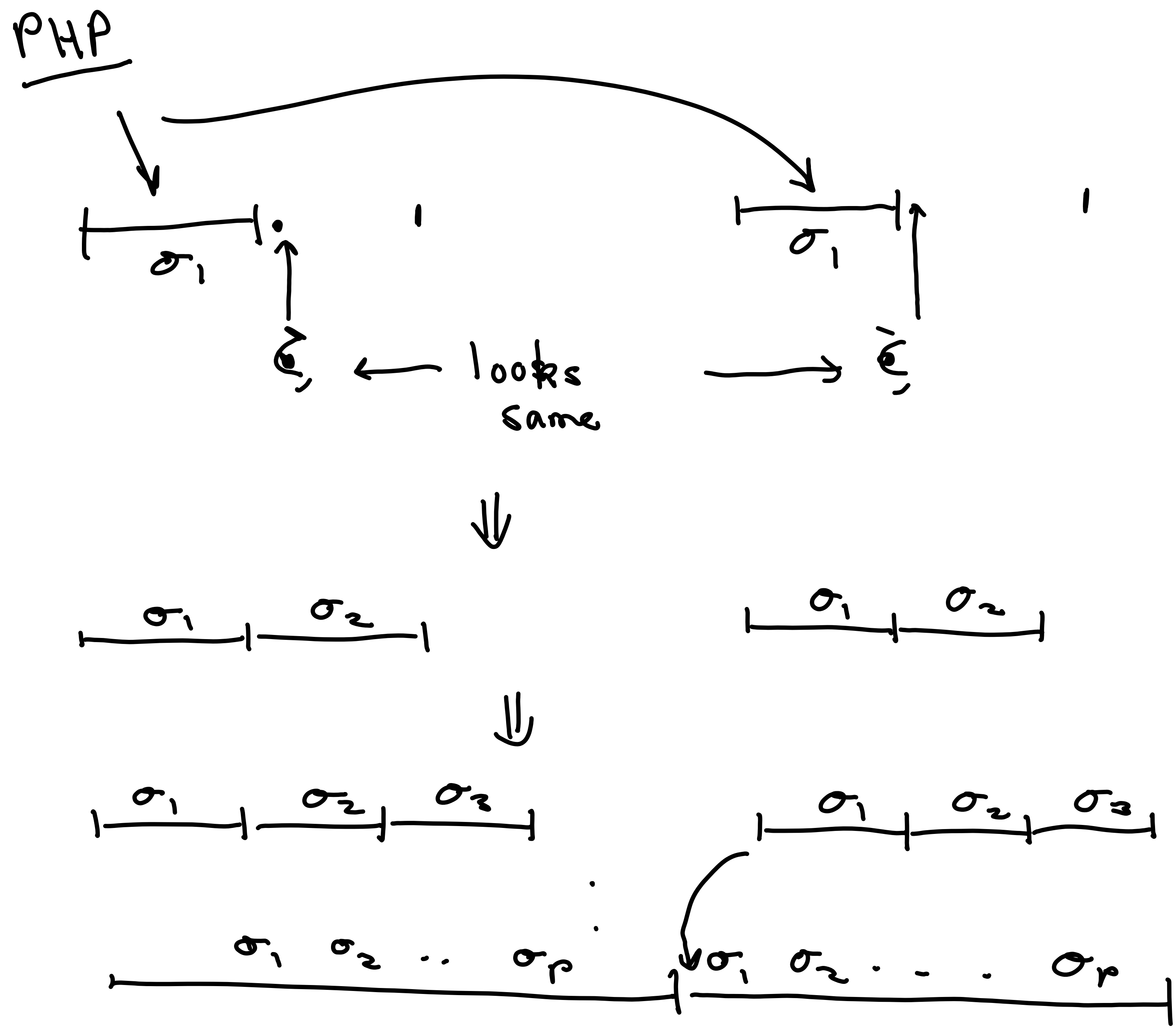
So

$$g(n) = \operatorname{mex}(n - S) \leq k$$

(ii)



possibilities $\leq k^a$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Case 2 $\quad |\bar{S}| < \infty$

g is ultimately arithmetic periodic if $\exists\, n_0, p, s$ such that if $n \geq n_0$ then $g(n+p) = g(n) + s$

Thm

$|\bar{S}| < \infty \Rightarrow$ g is ultimately arithmetic periodic.

Proof

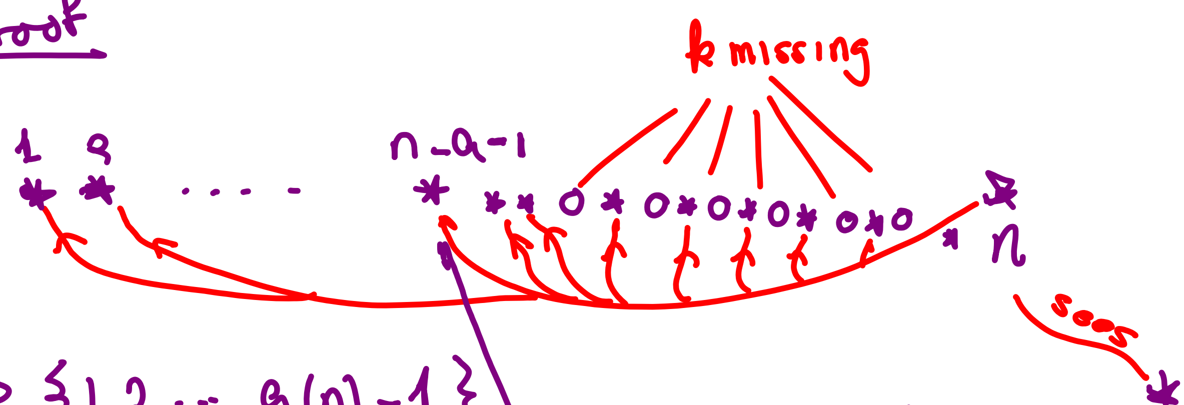
Let $S=\{a_1<a_2<\cdots<a_k=a\}$ as before.

Claim 1

If $n \geq a$ then

$$k-a \leq g(n+1)-g(n) \leq a-k+1.$$

Proof



$\{*\} \supseteq \{1,2,\dots,g(n)-1\}$

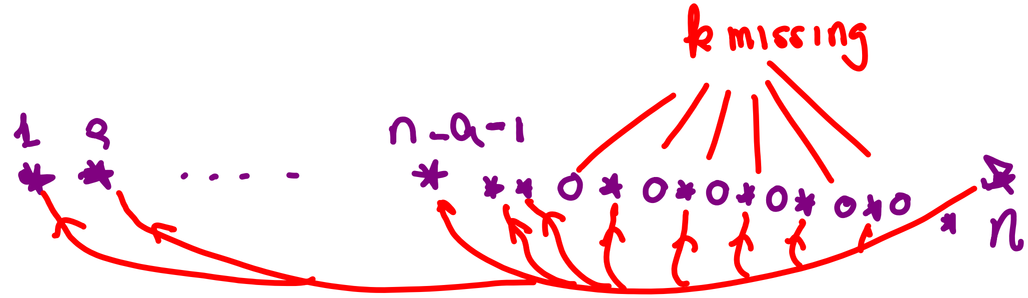
$\exists\, m \leq n-a-1$ s.t.

$g(m) \geq g(n)-1-(a-k)$

$\leq m$ are options for $n+1$

$\Rightarrow g(n+1) > g(m)$

$\Rightarrow g(n+1)-g(n) \geq k-a$



similarly $\exists\, m \leq n-a-1$ such that

$$g(n) \geq g(m) \geq g(n+1) - 2 - (a-k)$$

$\square$

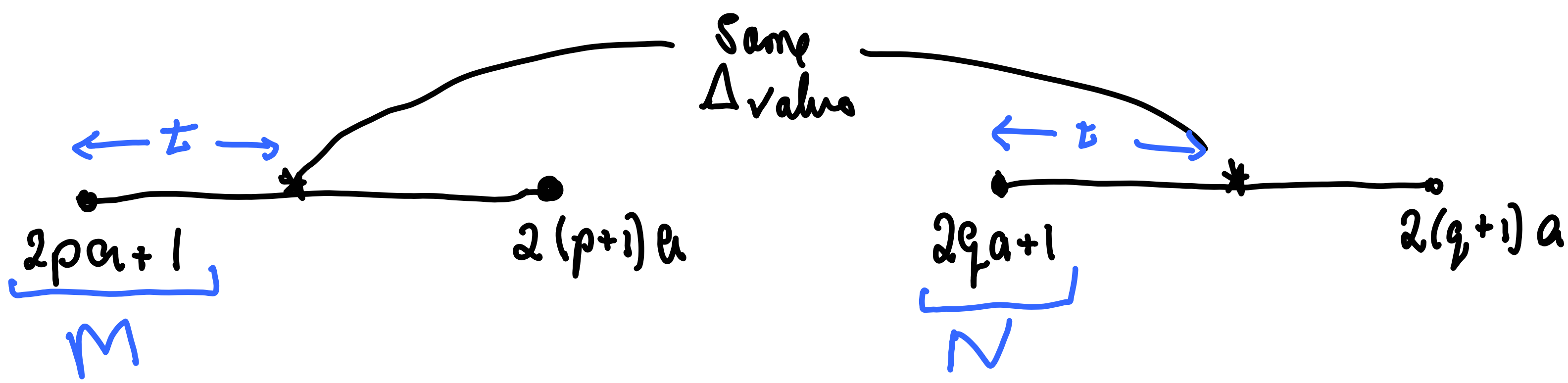
So

$$|g(n+1) - g(n)| \leq a-k+1$$

$$\Delta_n = g(n+1) - g(n)$$

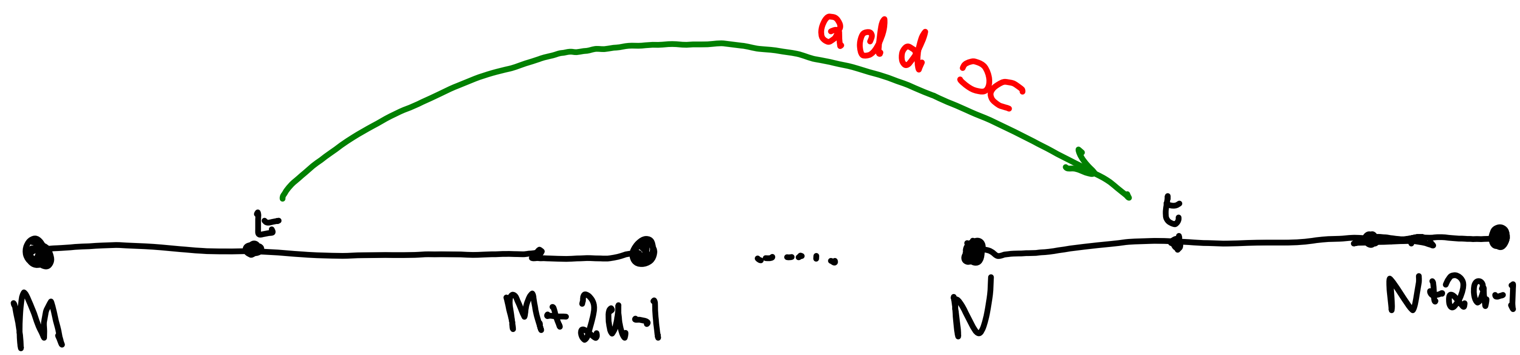
$$\#\text{ choices for sequence} \leq [2(a-b)+3]^{2a}$$

Whole $2a$ length Δ sequence must repeat.



$$\forall t \quad g(M+t+1) - g(M+t) = g(N+t+1) - g(N+t)$$

$$\forall t \quad g(N+t+1) - g(M+t+1) = g(N+t) - g(M+t) = x$$



Only need to check that $g(N+2a) = g(M+2a) + S$ and use induction.

$P = M$ or N

$$g(P+2a) = \text{mex}\left(g(P+2a-S)\right)$$

$$= \text{mex}\left(\underbrace{g(0), g(1), \ldots, g(\;+a)}, g([P+a+1, P+2a]-S)\right)$$

But $g(N+2a) >$

and $g([N+a+1, N+2a] - S)$

$= g([M+a+1, M-2a] - S) + x.$

$\square$