Maker Breaker Connectivity

Maker and Breaker play alternatively coloring edges of the complete graph.

Maker chooses 1 uncolored edge and colors it Mauve

Breaker chooses b uncolored edges and colors them Blue

Maker wins if the mauve graph contains a spanning tree, otherwise Breaker wins

Theorem 1 (Gebauer, Szabo)

If $b \leq b_0 = \frac{n}{\ln n}\left(1 - \frac{\ln\ln n}{\ln^2 n} - \frac{6}{\ln^2 n}\right)$

then Maker has a winning strategy.

Proof

We assume that Breaker goes first.

$B_1 \quad M_1 \quad B_2 \quad M_2 \quad B_3 \quad M_3 \quad \cdots \quad B_i \quad M_i$

$d(v)$ = degree of v in $\mathcal{B}$ ← blue graph.

$C(v)$ = component of $\mathcal{M}_t$ ← mauve graph.
containing v.

A component $C(v)$ is dangerous if $|C_1(v)| \leq 2b$

danger of v = $\psi(v) = \begin{cases} d_t(v) & \text{if } C_t(v) \text{ is dangerous} \\ -1 & \text{otherwise} \end{cases}$

Initially, every vertex is <u>active</u>

Maker's strategy: step M_t.

Choose v_t, an active vertex v with largest value of $\psi(v)$.

$v_t w$ is an arbitrary edge e joining $C_1(v_t)$ to $C(v_t)$. Color e mauve.

Re-activate v_t.

Observation: Each component of $\mathcal{M}$ has a unique active vertex.

Suppose now that Breaker has a winning strategy.

Let g be first time that Breaker colors all the edges of some cut $(K : \bar{K})$ blue.

$g \leq n-1$ because first $t \leq n-1$ mauve edges form a forest.

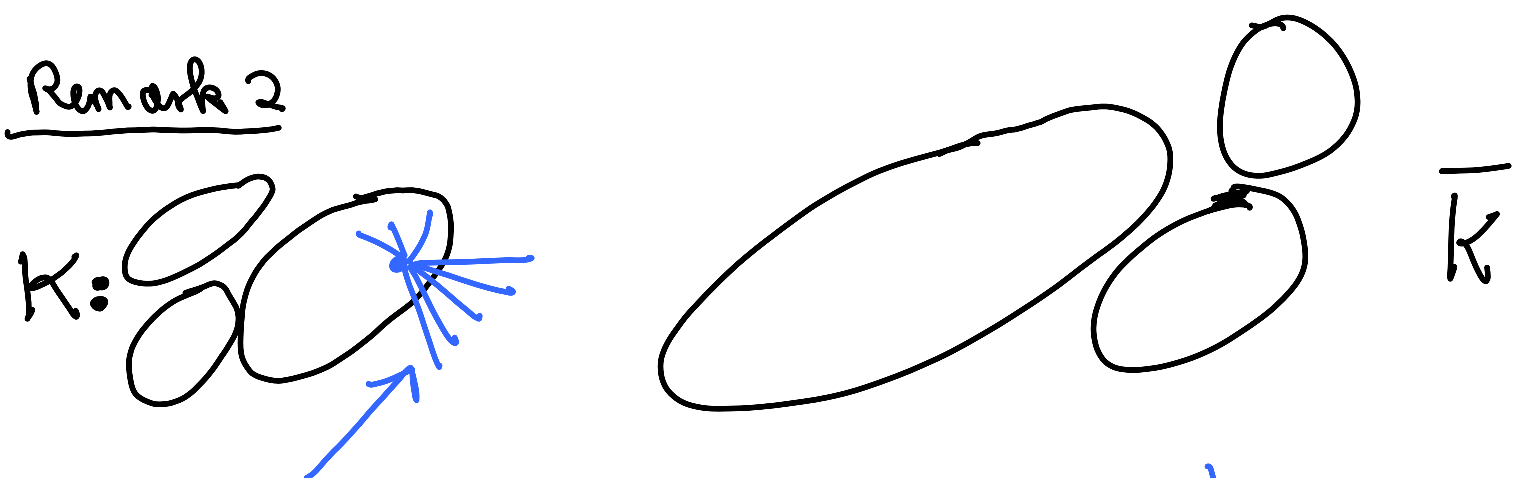
Assume $|\bar{K}| \leq |\bar{K}|$.

$|K| < 2b$ else $|K : \bar{K}| \geq 2b(n-2b) > bn$.

Remark 1

v_i is in a dangerous component up to its deactivation.

Remark 2



degree $\geq n - 3b$ just before B's last move.

K

v_g = arbitrary active vertex at time of B's last move.

$v_1, v_2, \dots, v_{g-1}$ are defined in game.

$$I_t = \{ v_{g-t}, \dots, v_g \}$$

For $I \subseteq [n]$ we let

$$\overline{\Psi}_{B,t}(I) = \frac{1}{|I|} \sum_{v \in I} \psi(v)$$

just before B_t

M M_t

Lemma 1

$1 \leq t \leq g-1$ implies

$$\overline{\psi}_{M,g-t}(I_t) \geq \overline{\psi}_{B,g-t+1}(I_{t-1})$$

Proof

M_{g-t}

$v_{g-t}, v_{g-t+1}, v_{g-t+2}, \ldots, v_g$ $\quad I_t$

$\longleftarrow$ dangerous at this time $\longrightarrow$

B_{g-t+1}

$v_{g-t+1}, v_{g-t+2}, \ldots, v_g$ $\quad I_{t-1}$

$\therefore$

$$\overline{\psi}_{M,g-t}(I_{t-1}) = \overline{\psi}_{B,g-t+1}(I_{t-1})$$

$\leq \overline{\psi}_{M,g-t}$ by choice of v_{g-t} (most dangerous).

Lemma 2

(a) $$\overline{\Psi}_{M,g-t}(I_t) - \overline{\Psi}_{B,g-t}(I_t) \leq \frac{2b}{t+1}.$$

(b) $$\overline{\Psi}_{M,g-t}(I_t) - \overline{\Psi}_{B,g-t}(I_t) \leq \frac{b+t+a(t-1)-a(t)}{t+1}$$

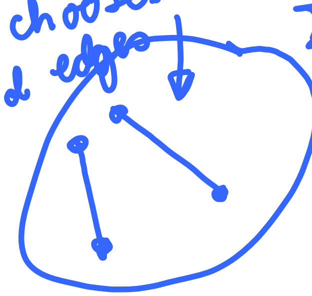
where $a(t) = \#$ edges spanned by I_t that B took in rounds $1, 2, \dots, g-t-1$.

Proof

(a) B move does not change $C(v)$ — effect on danger

B move only affects $d(v)$

Adds at most $b + e_d \leq 2b$ to sum of degrees in I_t



(b)

$$a(t) + e_d \leq a(t-1) + t$$

blue edges $\subseteq I_t$ chosen in rounds $1, \ldots, g-t$

blue edges $\subseteq I_{t-1}$ chosen in rounds $1, \ldots, g-t$

$\geq$ # edges $\subseteq I_t$ incident to v_{g-t}

$\square$

We show $\Psi_{B,1}(I_{g-1}) > 0$ — contradiction.
should be 0.

$$k = \left\lfloor \frac{n}{\ln n} \right\rfloor$$

$g < k$ and $g \geq k$ treated seperately

$\underline{g < k}$

$$\overline{\Psi}_{B,1}(I_{g-1}) = \overbrace{\overline{\Psi}_{B,g}(I_0)}^{\geq n-3b} + \overbrace{\sum_{t=1}^{g-1}\left(\overline{\Psi}_{M,g-t}(I_t) - \overline{\Psi}_{B,g-t+1}(I_{t-1})\right)}^{\geq 0}$$

$$- \sum_{t=1}^{g-1}\left(\overline{\Psi}_{M,g-t}(I_t) - \overline{\Psi}_{B,g-t}(I_t)\right)$$

$$\geq n-3b - \sum_{t=1}^{g-1} \frac{b+t+a(t-1)-a(t)}{t+1}$$

$$\geq n-3b - b(H_g - 1) - \frac{a(0)}{2} + \sum_{t=1}^{g-2} \frac{a(t)}{(t+2)(t+1)} + \frac{a(g-1)}{g}$$

$$\geq n - b(H_g + 2) - g$$

$$\geq n - b(\ln k + 3) - k$$

$$\geq n - \frac{n}{\ln n}(\ln n - \ln\ln n + 3) - \frac{n}{\ln n} > 0.$$

$\underline{g \geq k}$

$$\overline{\Psi}_{B,1}(I_{g-1}) = \underbrace{\overline{\Psi}_{B,g}(I_0)}_{\geq n-3b} + \underbrace{\sum_{t=1}^{g-1}\left(\overline{\Psi}_{M,g-t}(I_t) - \overline{\Psi}_{B,g-t+1}(I_{t-1})\right)}_{\geq 0}$$

$$- \sum_{t=1}^{k-1}\left(\overline{\Psi}_{M,g-t}(I_t) - \overline{\Psi}_{B,g-t}(I_t)\right)$$

$$- \sum_{t=k}^{g-1}\left(\overline{\Psi}_{M,g-t}(I_t) - \overline{\Psi}_{B,g-t}(I_t)\right)$$

$$\geq n-3b - \sum_{t=1}^{k-1}\frac{b+t+a(t-1)-a(t)}{t+1} - \sum_{t=k}^{g-1}\frac{2b}{t+1}$$

$$\geq n-3b - b(H_k-1) - (k-1) - \frac{a(0)}{2}$$

$$+ \sum_{t=1}^{k-2}\frac{a(t)}{(t+2)(t+1)} + \frac{a(k-1)}{k} - 2b(H_g - H_k)$$

$$\geq n - b(2H_g - H_k + 2) - k$$

$$\geq n - b(2H_g - H_k + 2) - k$$

$$\geq n - b(2\ln n - \ln k + 4) - k$$

$$\geq n - \left(\frac{n}{\ln n} - \frac{n \ln\ln n}{\ln^2 n} - \frac{6n}{\ln^2 n}\right)(\ln n + \ln\ln n + 5) - \frac{n}{\ln n}$$

$$\geq \frac{n(\ln\ln n)^2}{\ln^2 n}$$

$$> 0.$$

$\square$

Theorem 2 (Chvátal, Erdős)

If $b \geq b_1 = \frac{n}{\ln n}(1+\epsilon)$ — ϵ arbitrary constant

then Breaker has a winning strategy.

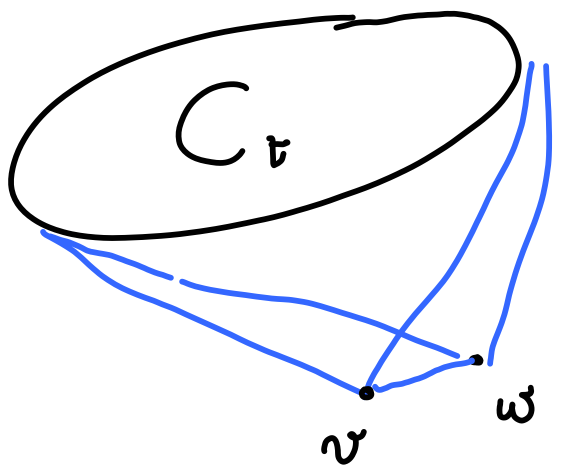
Proof

Let $k = \left\lceil \frac{n}{2\log n} \right\rceil$. Maker goes first.

After rounds, $1 \leq t \leq k$, Breaker can build a clique C_t of size t such that Maker has no edge incident with C_t.

$t = 1$ is trivial.

Suppose true for t.



Breaker move: claim vw plus all edges $\{v,w\} : C_t$

Maker's next move can only choose one edge incident with one vertex of $C_t \cup \{v,w\}$.

$C = C_k = \{1, 2, \dots, k\}$ $\quad A_i = \{(i,\ell) : \ell > k\}$

set of edges

Claim: Breaker can claim an A_i.

<u>Proof</u>

Define $f(1,b) = 0;$

$$f(k,b) = \lfloor k\,(f(k-1,b)+b)/(k-1) \rfloor$$

$$f(k,b) \geq (b-1)\,k \sum_{i=1}^{k-1} \frac{1}{i}$$ — easy induction on k.

Box Game: Disjoint sets $A_1, A_2, \dots, A_k$
(k,t) parameters
such that $\big|\,|A_i| - |A_j|\,\big| \leq 1, \ \forall i,j$

$t = |A_1| + |A_2| + \dots + |A_k|$

M can choose one element of one A_i

B can choose b elements from $\bigcup A_i$

B wins if he/she claims all elements of some A_i.

Claim B wins if $t \leq f(k,b)$. $f(2,b) = 2b$

Proof. Induction on k - $k=2$ trivial

After M move, B move: $k \to k-1$

lose one A_i

$t \to t^* \leq t - \lfloor t/k \rfloor - b$

$\leq f(k,b) - \lfloor t/k \rfloor - b \leq f(k-1, b)$. $\square$

For connectivity game, need only check that:

$$k(n-k) \leq (b-1)k \sum_{i=1}^{k-1} \frac{1}{i} \leq f(k,b).$$

#edges

$C : \overline{C}$

$\square$