Cops and Robbers

$G=(V,E)$ is a graph: $n=|V|$.

Step 0

(a) Place c cops at vertices of G.

(b) Place robber at some vertex of G

⋮

Step i

(a) Move some of the cops to adjacent vertices.
If cop occupies same vertex as robber, cops win.

(b) Move robber to adjacent vertex, or leaves it.

$c(G)$ = cop number

= minimum number of cops needed to garantee a win for the cops.

$c(G) \leq n$, clearly.

Conjecture: $c(G) = O(\sqrt{n})$, for all connected G.
(Meyniel)

If components of G are $G_1, G_2, \ldots$ then

$$c(G) = c(G_1) + c(G_2) + \cdots$$

Thm 1 $c(G) \leq \dfrac{n}{2^{(1-o(1))\sqrt{\log_2 n}}}$

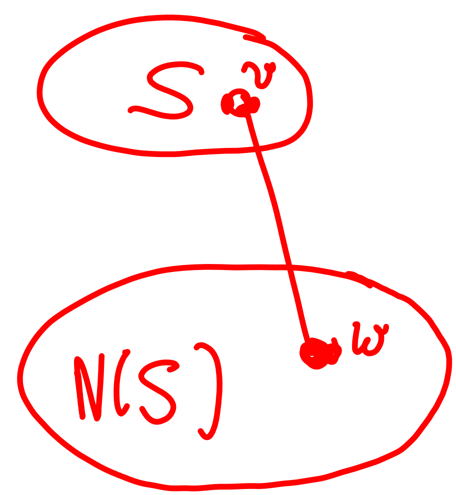
Proof

G is a p-expander, $0<p<1$ if

$$|N(S)| \geq p|S|$$

for all $S \subseteq V$, $|S| \leq n/2$.

Here $N(S) = \{ w \notin S : \exists v \in S \text{ s.t. } vw \in E(G) \}$



Lemma 1

Let $0 < p < 1$ be arbitrary.

Then
$$c(G) \leq np + c(G')$$
where

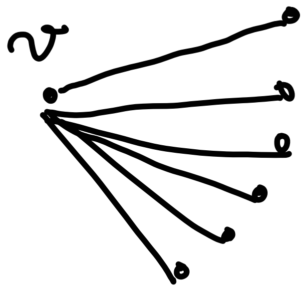
(a) G' has $m \leq n$ vertices

(b) G' has maximum degree $\leq 1/p$

(c) G' is a p-expander.

Proof of Lemma 1

Suppose v has degree $> 1/p$



A cop left at v will mean robber cannot use $N(v)$

$$c(G) \leq 1 + c(\underbrace{G - (v + N(v))}_{G_1})$$

Repeat until $\Delta(G_i) \leq \frac{1}{p}$.

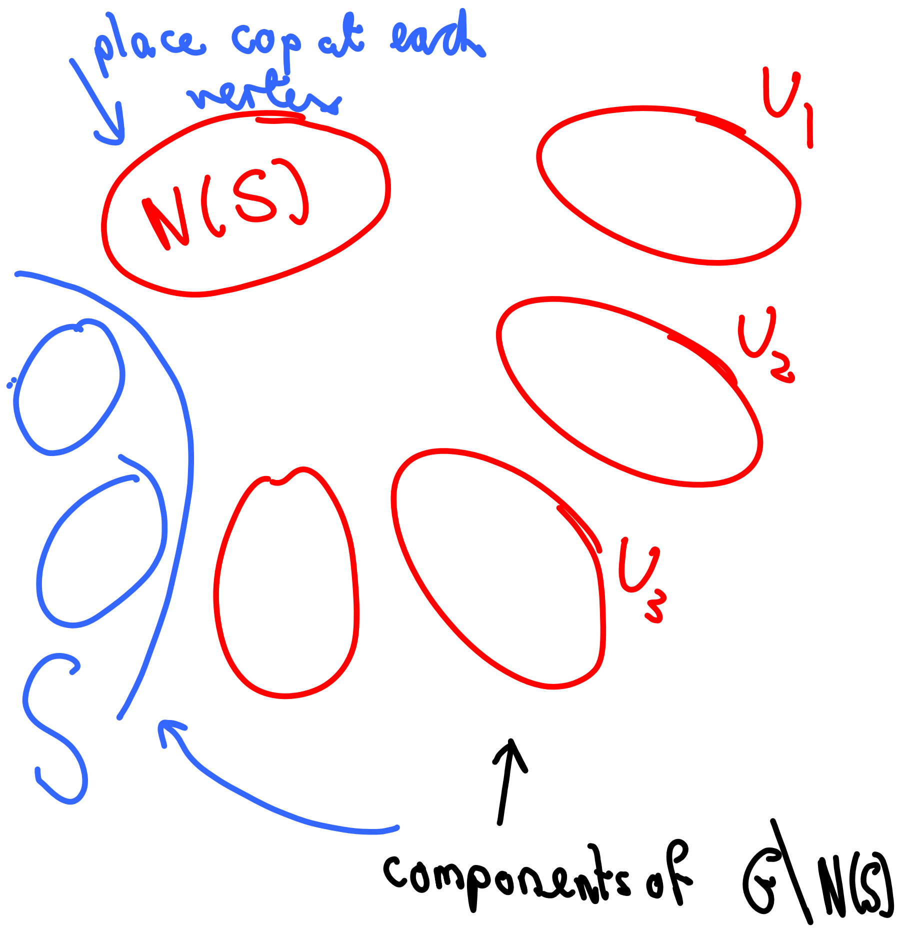
$$c(G) \leq i + c(G_i) \qquad (a)$$

$$|V(G_i)| \leq n - i\left(1 + \frac{1}{p}\right)$$

Suppose next that $\exists S,\ |S| < \frac{n}{2}$ and $|N(S)| < p|S|$.

Then

$$c(G) \le p|S| + c(G') \text{ where } |V(G')| \le n - |S|$$



$$c(G) \le |N(S)| + \max_i c(U_i)$$

Every component U spanned by S satisfies

$$|U| \le |S| < \frac{n}{2} < n - |S|$$

$$|U_i| \le n - |S| - |N(S)|.$$

Repeat until what is left is a p-expander.

$$c(G) \leq i + p\big(\underbrace{n - |N(G_i)|}_{\leq n - i(1 + 1/p)}\big) + c(G')$$

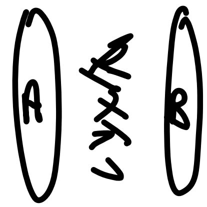
$\Delta \leq \frac{1}{p}$ & p-expander

$$\leq i + pn - ip - i + c(G')$$

$$= np + c(G').$$

$\square$

Lemma 2

Given a bipartite graph  and integer k

we can partition $A = S \cup T$ so that

(i) $|N(S)| \leqslant k|S|$

(ii) $|N(U)| > k|U|$ for all $U \subseteq T$.

Proof

Start with $S = \emptyset$ and $A^* = A$.

while $\exists U \subseteq A^*$ with $|N(U)| \leqslant k|U|$

$S \leftarrow S + U$; $A^* \leftarrow A - U$.

$\square$

Lemma 3

Let n, p, r be given np large. Then one can distribute $2pn$ cops so that

$$|B_r(S)| \geq \frac{16}{p}|S| \log n \Rightarrow B_r(S) \text{ contains } \geq |S| \text{ cops}$$

vertices with distance r of S.

<u>Proof</u> Place cops randomly by placing cop at vertex with probability p. Let $Z(S) = \#$ cops in $B_r(S)$.

$$Z(S) \geq Bin\left[\frac{16|S|}{p}, p\right]$$

$$\Pr(Z(S) \leq 8|S|) \leq e^{-\frac{1}{8}\cdot 16|S|\log n} = n^{-2|S|}$$

Chernoff Bound

$$\Pr(\exists S : Z(S) \leq 8|S|) \leq \sum_{s=1}^{n} \binom{n}{s} n^{-2s} = o(1)$$

$$\Pr(\#\text{cops} \geq 2np) \leq e^{-np/3}.$$

$\square$

Lemma 4

$\exists\ p = 2^{-(1-o(1))\sqrt{\log_2 n}}$ such that if

$$k = \frac{16}{p}\log n \quad \& \quad \ell = (1+o(1))\sqrt{\log_2 n}$$

then

$$k^{\ell+1} \leq (1+p)^{-2^{\ell}}\, np/2 \qquad \text{(A)}$$

$$(1+p)^{2^{\ell}} \geq k. \qquad \text{(B)}$$

Proof

$$\log\frac{1}{p} = \Theta\left(\sqrt{\log_2 n}\right) \Rightarrow \log k = (1+o(1))\log\frac{1}{p}$$

$$\underbrace{\ell = \left\lceil \log_2 \left(\frac{\log k}{\log(1+p)} \right) \right\rceil}_{(1+p)^{2^\ell} \geq k \;\equiv\; B} = (1+o(1)) \log_2 \frac{1}{p}$$

$$k^{\ell+1} (1+p)^{2^\ell} \leq k^{\ell+1} k^2 = k^{\ell+3}$$

So to prove A we need to show that

$$(\ell+3) \log_2 k \leq \log_2 \frac{np}{2}$$

$(1+o(1)) \log_2 \frac{1}{p}$ $\qquad$ $(1-o(1)) \log_2 n$

$\square$

Proof of Theorem 1 (P3)

Assume G' has $n \geq np$ vertices. [$G' = G$ after removing high degree vertices etc.]

Cops: C_0 C_1 ..... C_i ..... C_ℓ

$2np$

$r = 2^i$

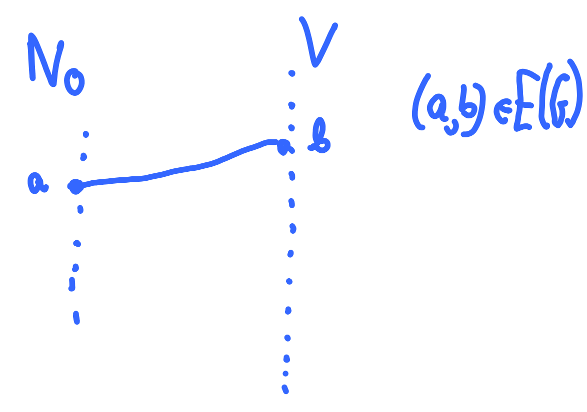
Lemma 3
(P9)

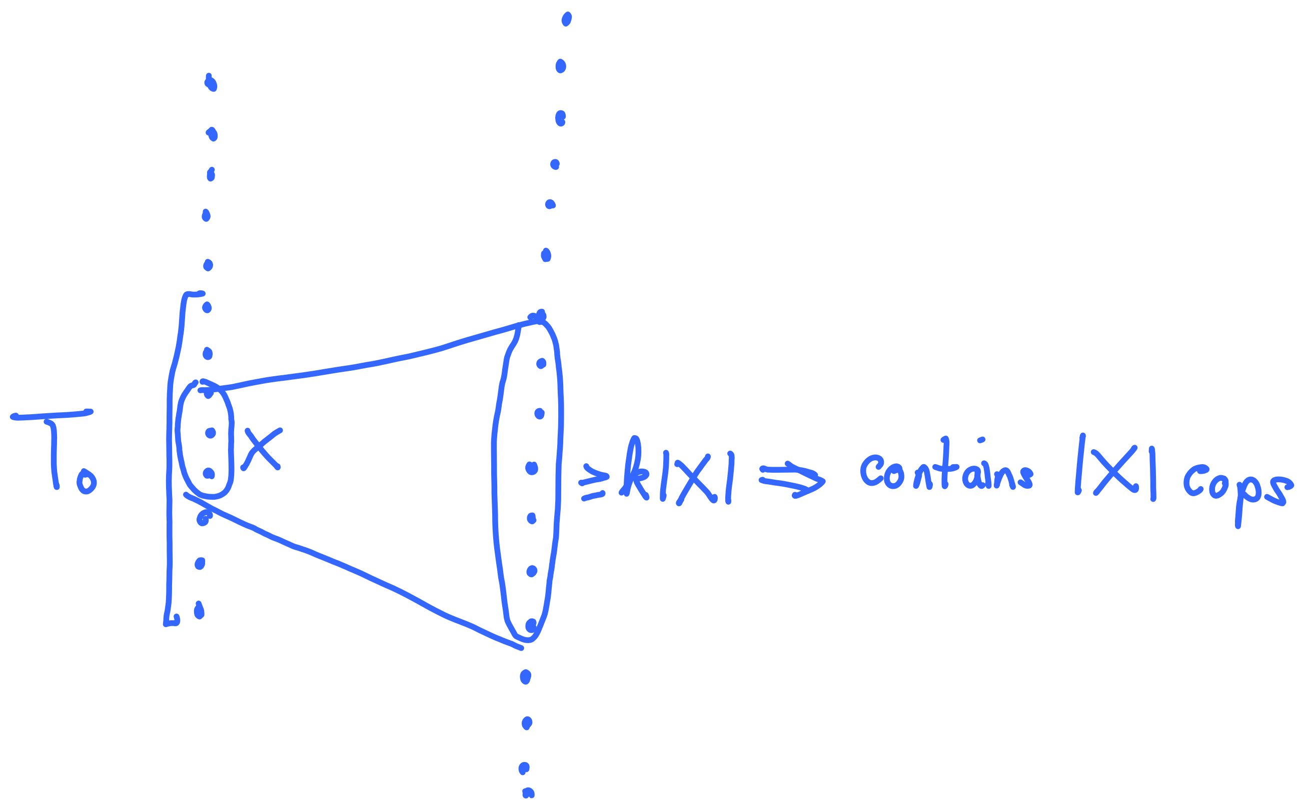
Robber starts at v:

$N_0 = N(v)$. $|N_0| \leq \frac{1}{p} < k = \frac{16}{p}\log n$

Lemma 2: $N_0 = S_0 \cup T_0$
(P8)

Bipartite Graph B_1





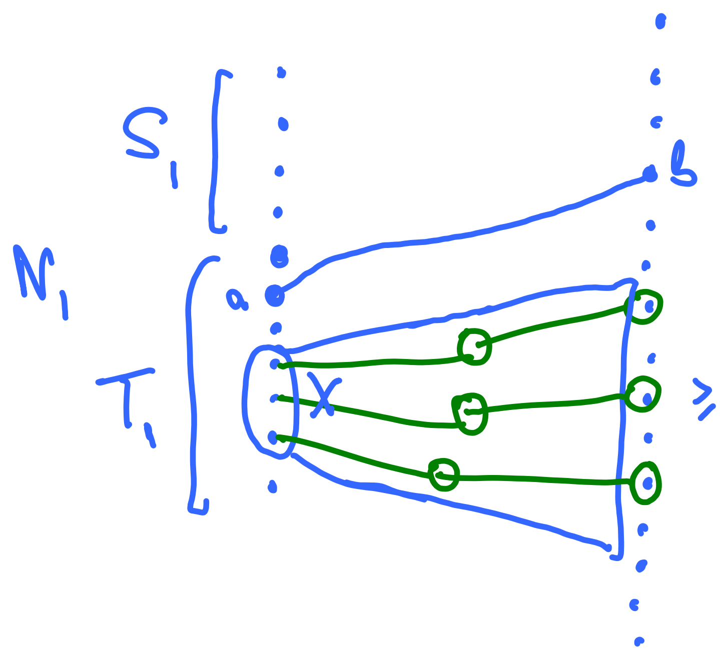
Apply Hall's Theorem: assign one cop for every vertex of T_0.

So robber cannot go to T_0 in move 1.

Robbers position after second move is restricted to $N_1 = B_1(S_0)$. $|N_1| \leq k|S_0| \leq k^2$

Partition $N_1 = S_1 \cup T_1$.

Cops in C_1 prevent robber from moving to T_1

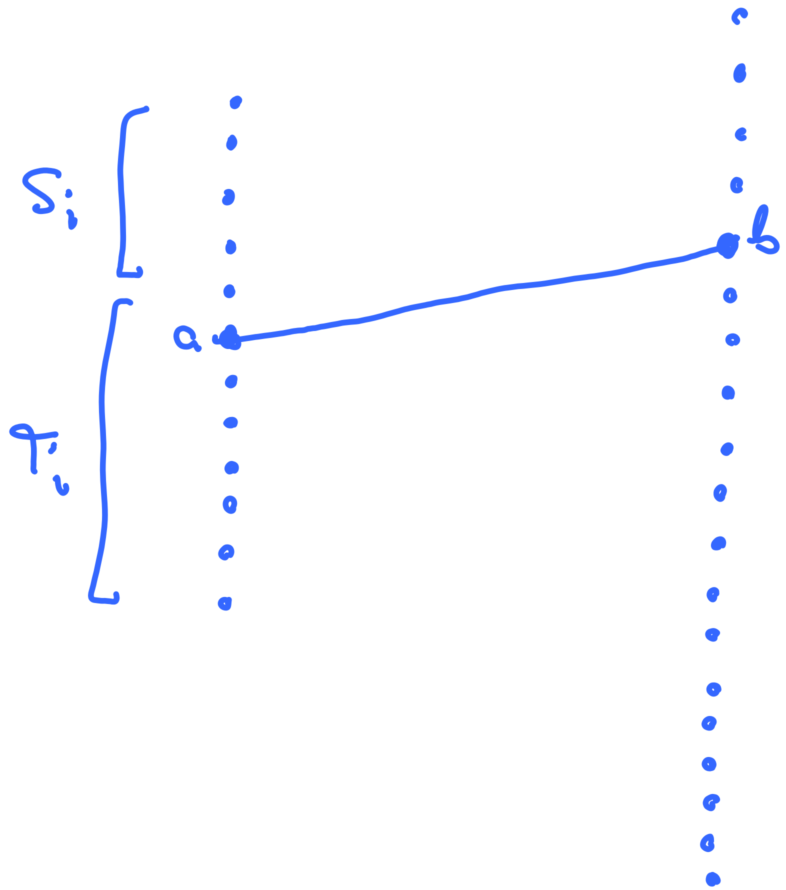


$\text{dist}(a,b) \leq 2$

$\geq k|X| \Rightarrow \geq |X|$ cops – move to cover T_1

So Robber's position after 4 moves restricted to $B_2(S_1) \leftarrow N_2$

In general after 2^i steps robber in $N_i = B_{2^{i-1}}(S_{i-1})$.



$dist(a,b) \leq 2^i$

Cops in C_i can prevent v from moving to T_i.

One can assign one cop to reach each $x \in T_i$ in 2^i steps.

Repeating, we see that after Robber's 2^i more he is restricted to $N_i := B_{2^{i-1}}(S_{i-1})$ of size $\leq k^i$.

Claim: $N_\ell = S_\ell \cup T_\ell$

$\uparrow$

$\emptyset$

Proof

$$|N_\ell| \leq k^{\ell+1} \leq (1+p)^{-2^\ell} np/2.$$

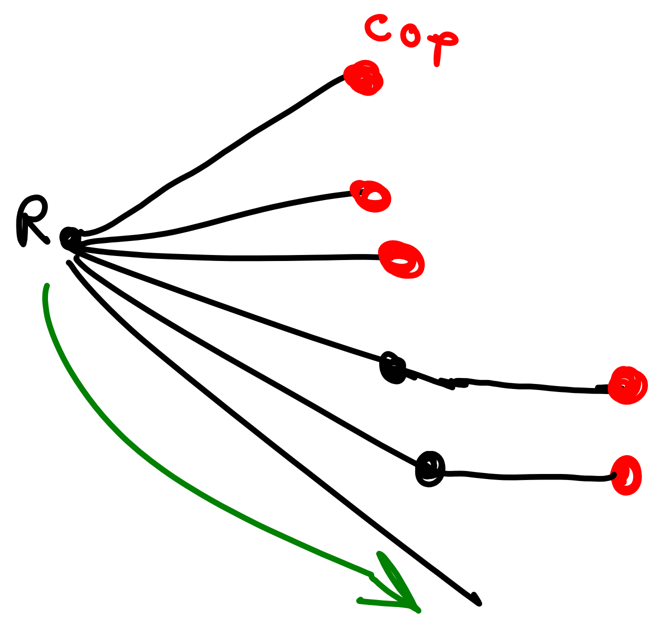
Every set $|X| \leq (1+p)^{-(2^\ell-1)} \frac{n}{2}$ has $|B_{2^\ell}(X)| \geq (1+p)^{2^\ell} |X| \geq k|X|$ $\Rightarrow S_\ell = \emptyset$

$\square$

Thm

Suppose that G is a d-regular graph with no C_3, C_4. Then $c(G) \geq d$.

Proof



Example

Projective plane

n vertices

$d \sim \sqrt{n}$

Theorem

If G is planar then $c(G) \leq 3$.

Proof

Lemma

If P is a shortest path from u to v in a graph H (not necessarily planar) then one cop can control the vertices of P.

Proof of lemma

Suppose at some time we have $\underset{\text{cop vertex}}{c} \in P$ and

$$d(\underset{\text{robber vertex}}{r}, z) \geq d(c, z) \quad \text{for all } z \in P \qquad (*)$$

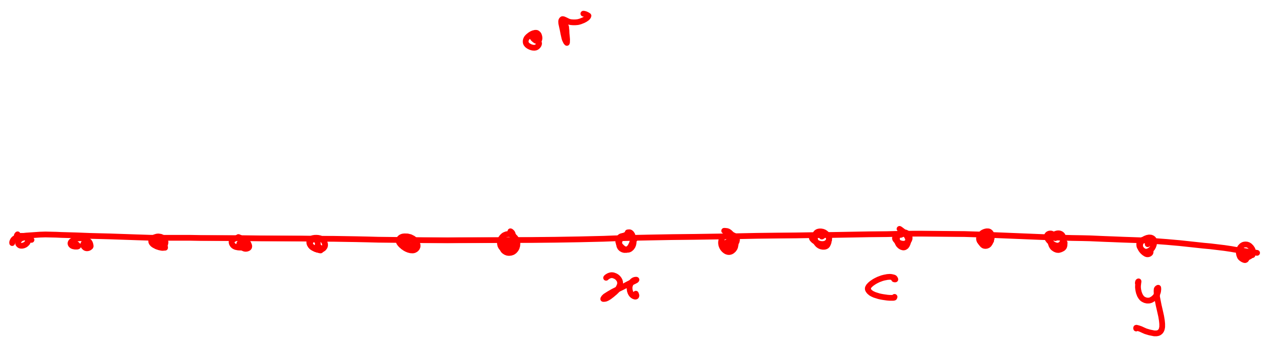
We show

(i) c can attain $(*)$

(ii) c can maintain $(*)$

In which case, once $(*)$ holds, robber cannot enter P.

Attaining (*)

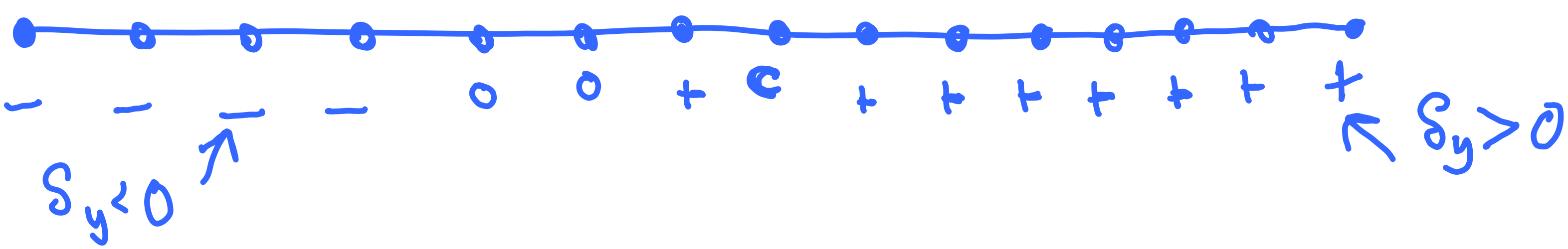


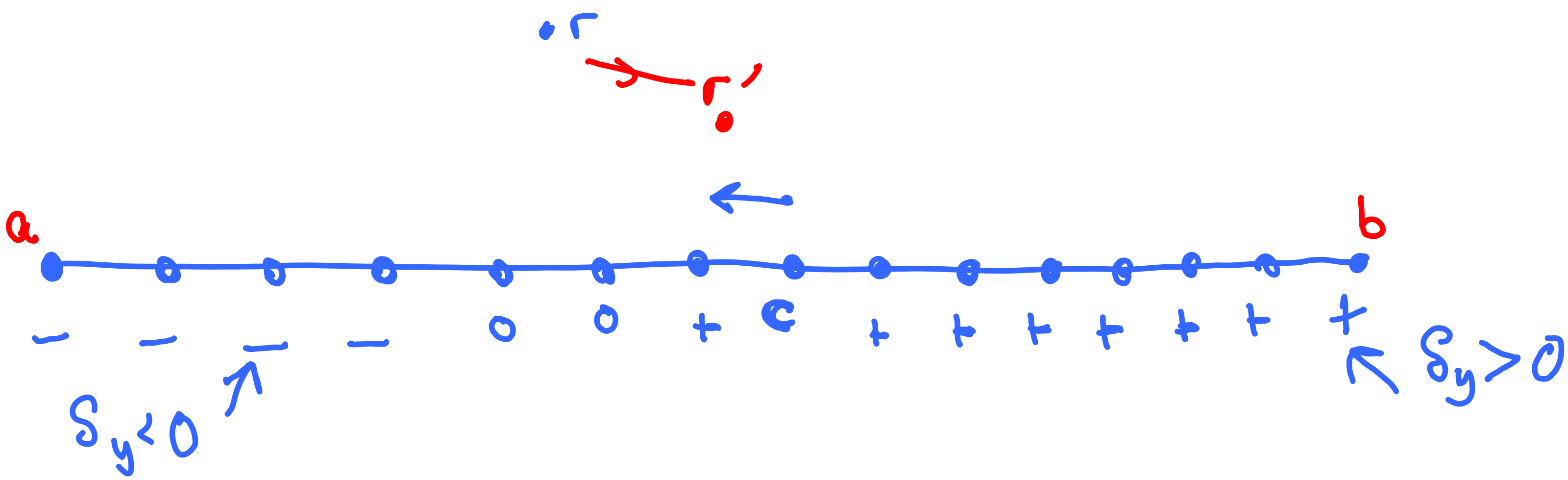
$$d(x,y) = d(x,c) + d(c,y)$$

$$d(x,y) \le d(x,r) + d(r,y)$$

So $\left[d(x,r) - d(x,c)\right] + [\underbrace{d(y,r) - d(y,c)}_{\delta_y = \delta_y(c,r)}] \geq 0$

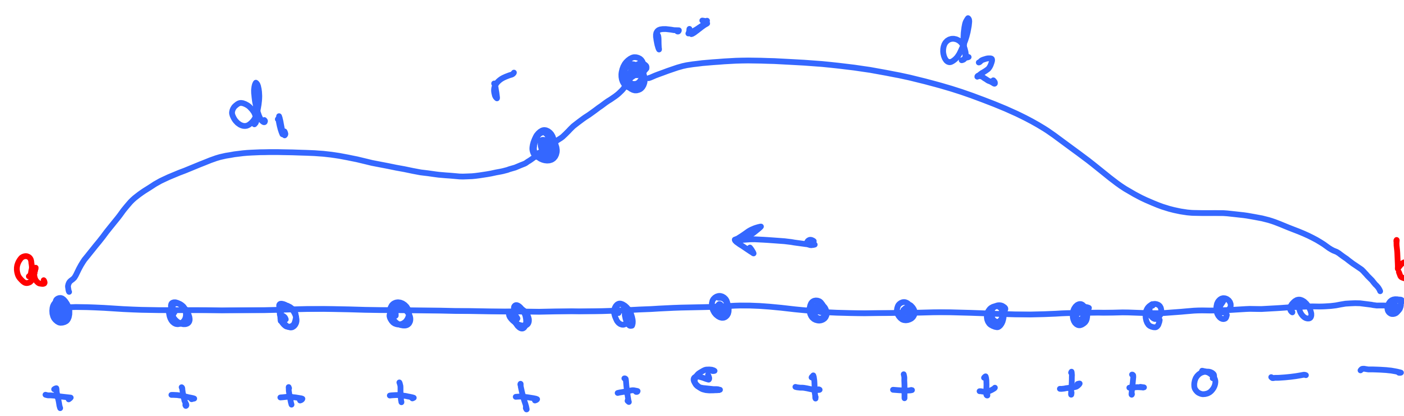
So for any c, r path looks like

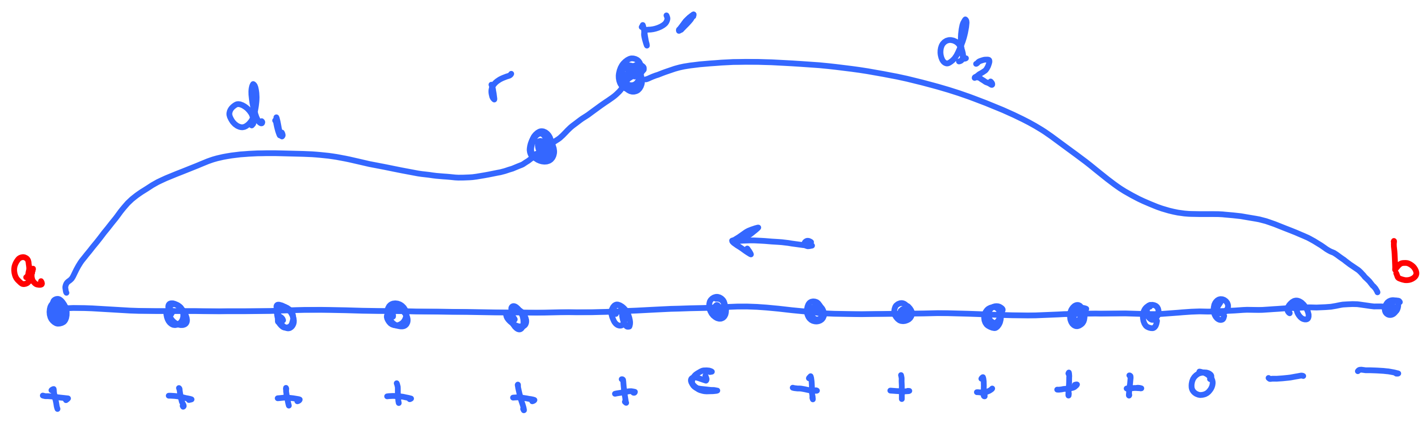




Cop moves towards end a where $\delta < 0$.

Suppose that robber moves to r'. If $\delta_b' \geq 0$ we are fine. Otherwise there could be a 'flip'





$$d_1 + 1 + d_2 \geq d(a,b)$$

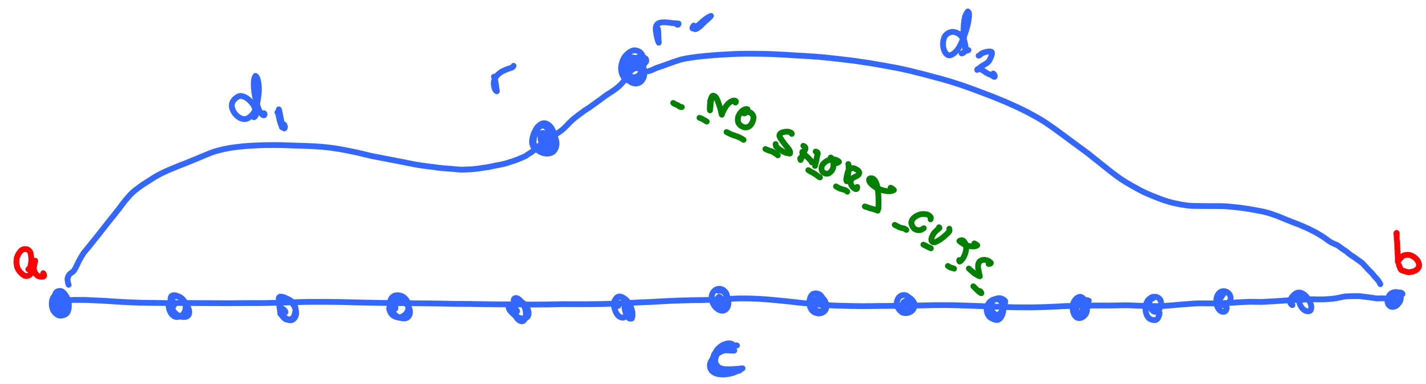
$$d(r,a) > d(c,a)$$

$$d(r',b) > d(c',b) = d(c,b) + 1$$

$d(c,a) \geq d_1 + 1$ & $d(c,b) \geq d_2 \Rightarrow$ $d(c,a) = d_1 + 1$, $d(c,b) = d_2$

So top path is also shortest.

Cop moves back from c' to c.



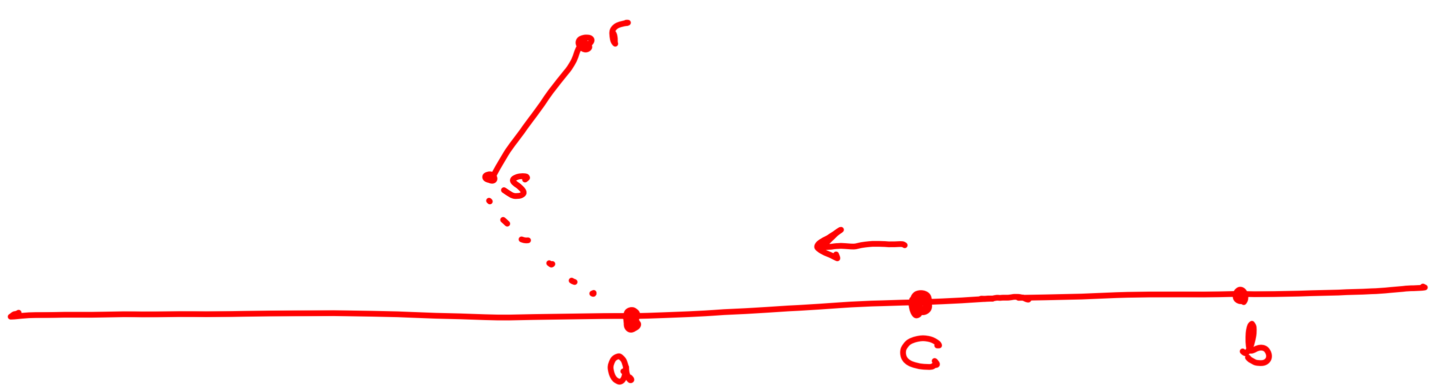
Now $d(a, r') = d_1 + 1 = d(a, c)$

and then (*) holds.

(ii) Maintenance of (*).

If R stays put, then C stays put.

Suppose R goes from r to s.



Suppose $d(a,s) = d(a,r) - 1 = d(a,c) - 1$

and $d(b,s) = d(b,c)$

then

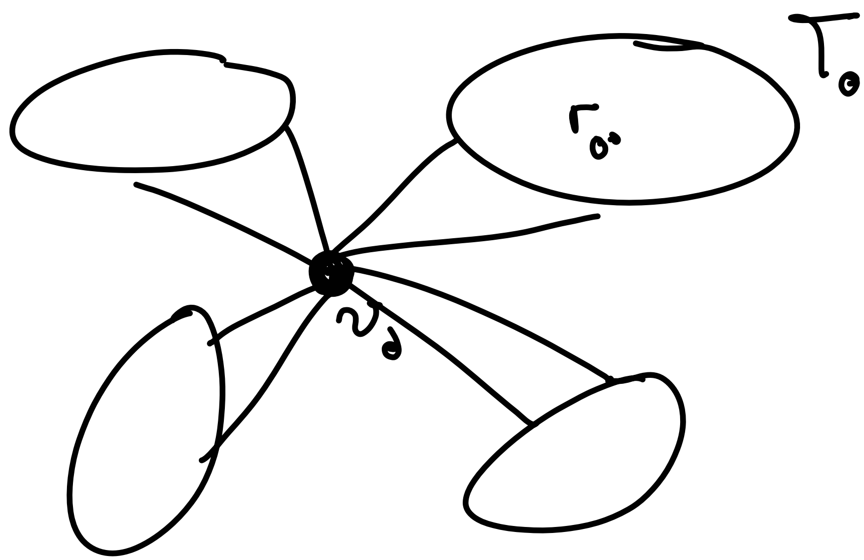
$$d(a,b) \le d(a,s) + d(b,s) = d(a,c) - 1 + d(c,b)$$
$$= d(a,b) - 1.$$ Contradiction.

Back to proof of theorem

At start cops occupy vertex v_0.

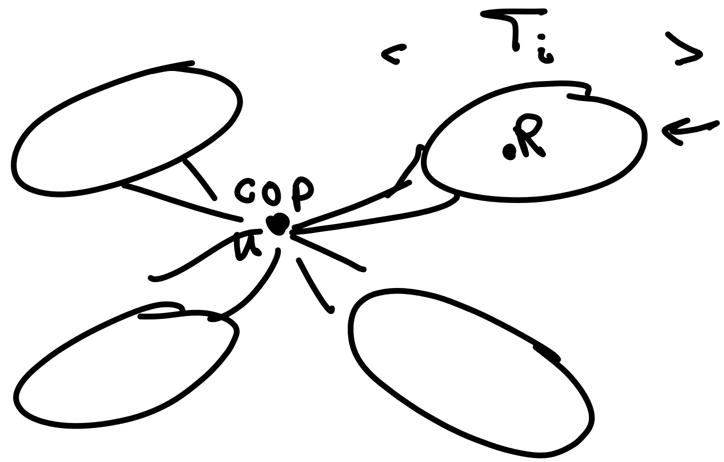
R moves to r_0.

Robber territory T_0 = component of $G - v_0$ containing v_0



General Stage i : 2 cases.

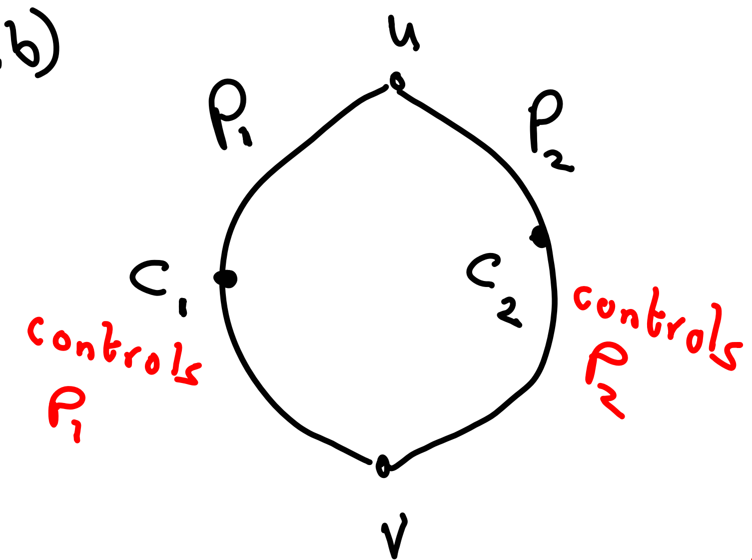
(a)



component of $G-u$ containing R

Starts in (a)

(b)



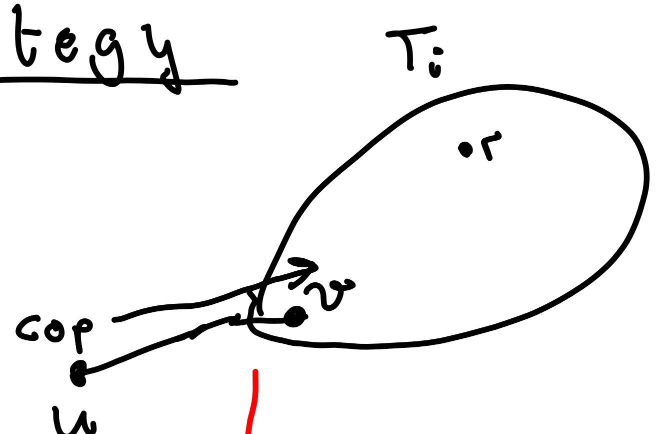
R in exterior $= T_i$

$|P_1| = d(u,v)$; P_2 = shortest $u \to v$ disjoint P_1.

Cop strategy

(a)

(i)



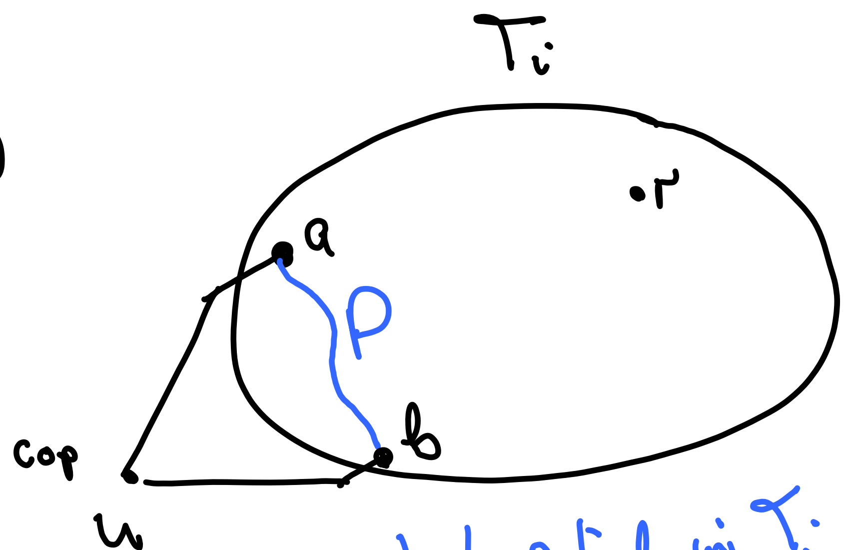
unique nbr in T_i

Back to (a)

v replaces u.

T_i shrinks

(ii)



P shortest a to b in T_i

Free cop moves to control P.

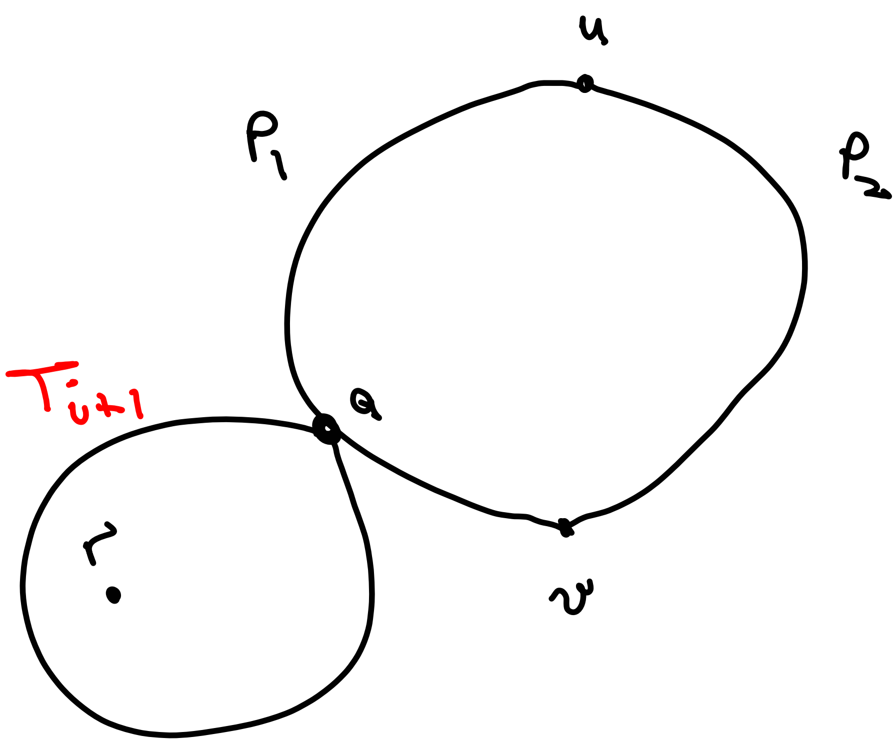
(i) $(a,b) \notin E$: $P_1 \leftarrow (a,u,b)$, $P_2 \leftarrow P$

(ii) $(a,b) \in E$: $P_1 \leftarrow P = (a,b)$, $P_2 \leftarrow (a,u,b)$

Case (b): $T_{i+1} = T_i \setminus P$

(b)

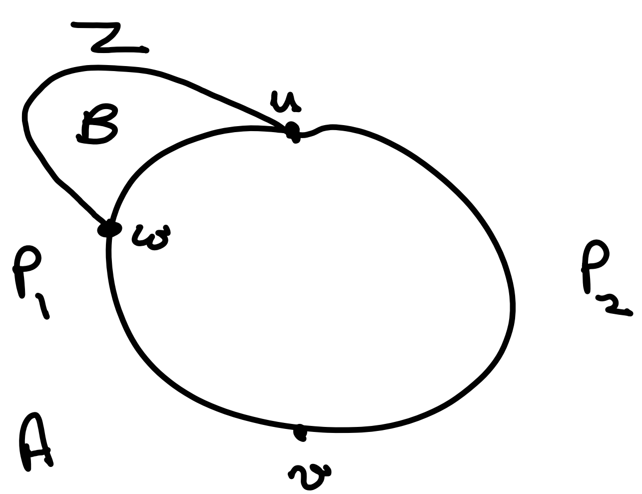
(1) No path in $T_i \cup P_1 \cup P_2$ from u to v other than P_1, P_2.

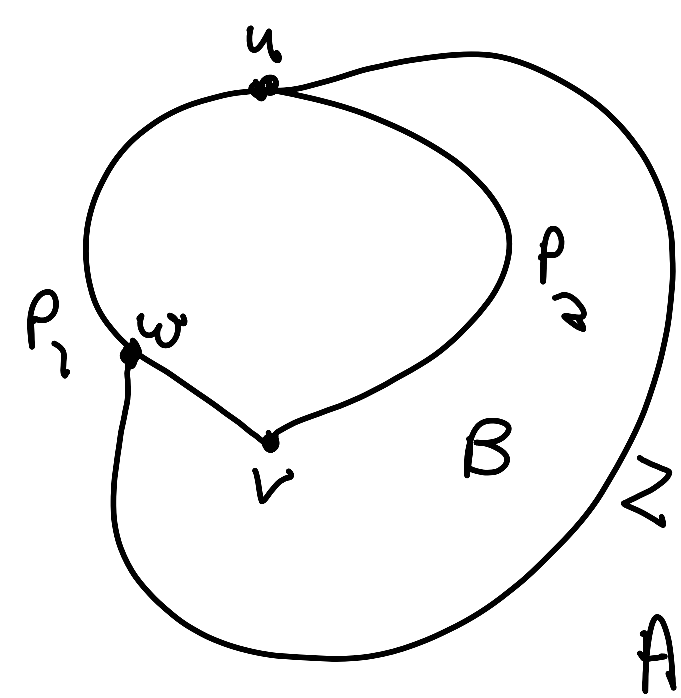


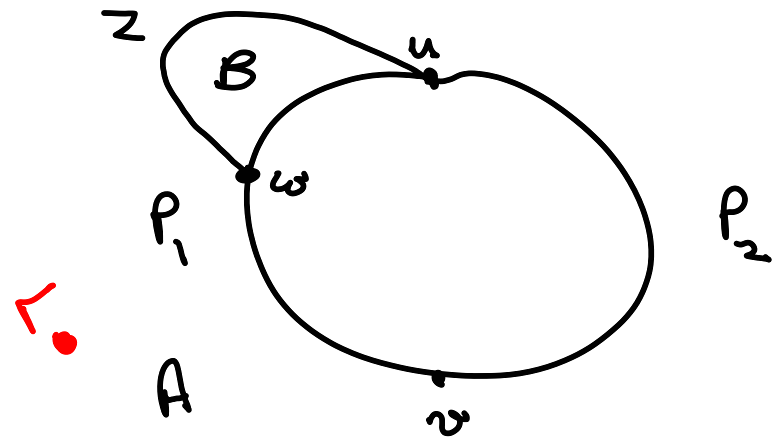
Free cop moves to a, back in Case (a).
Smaller T_i.

(ii) $\exists$ other u,v paths. Let Q be shortest such path in $T_i \cup P_1 \cup P_2$.
w is first vertex after u on Q.

(a) $w \in P_1$





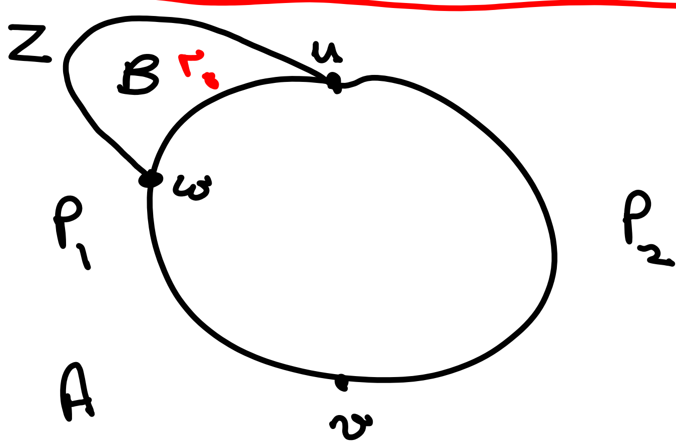


$P_3 = Z(u,w) + P_1(w,v)$
shortest path u to v, after $P_1(u,w)$ is deleted.

Free cop moves to control P_3.
Cop2 still controls P_2.

Now in Case (b)

T_i shrinks

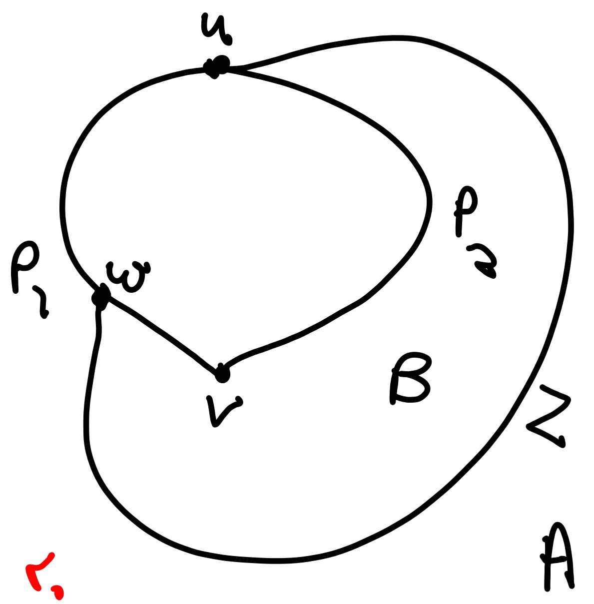


$Z(u,w)$ is shortest u to w avoiding $P_1(u,w)$.

Free cop moves to control $Z(u,w)$

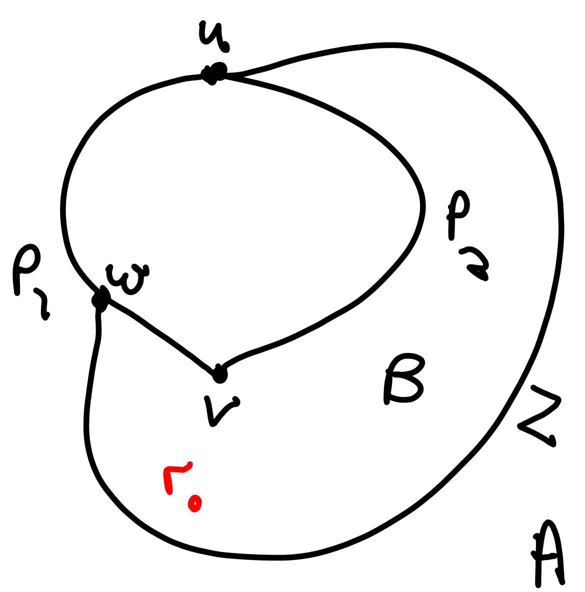
C_1 controls $P_1(u,w)$

T_i shrinks



Free cop moves to control $Z(u,w)$

C_1 now controls $P_1(u,w)$.

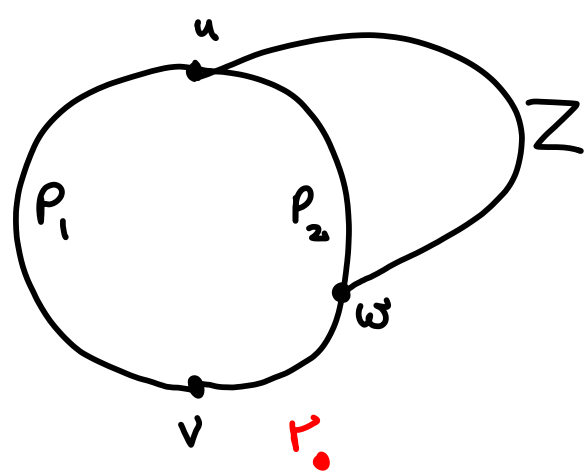


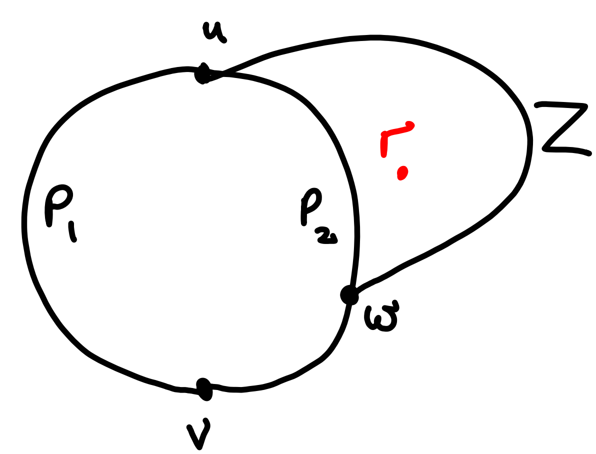
Free cop moves to control $Z(u,w)+P_1(w,v)$

C_2 remains in control of P_2.

(b) $w \in P_2$

(1) Z does not meet P_1, except at v.





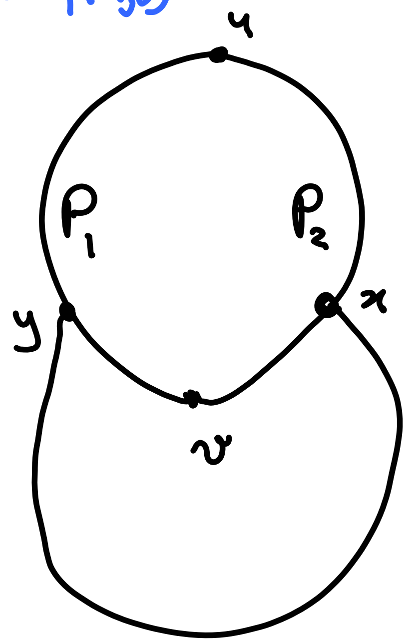
$P_3 = Z(u,w) + P_2(w,v)$ is shortest u to v avoiding P_1 (without using $P_2(u,w)$ interior)

(ii)

Z meets P_1.

Free copy occupies $P_2(u,x) + Z(x,y)$

C_i occupies $P_1(u,y)$

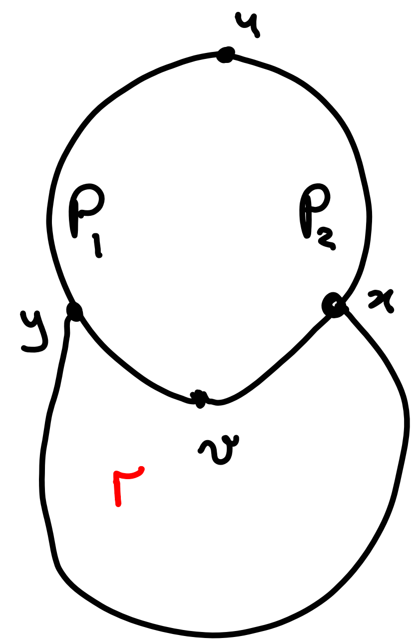


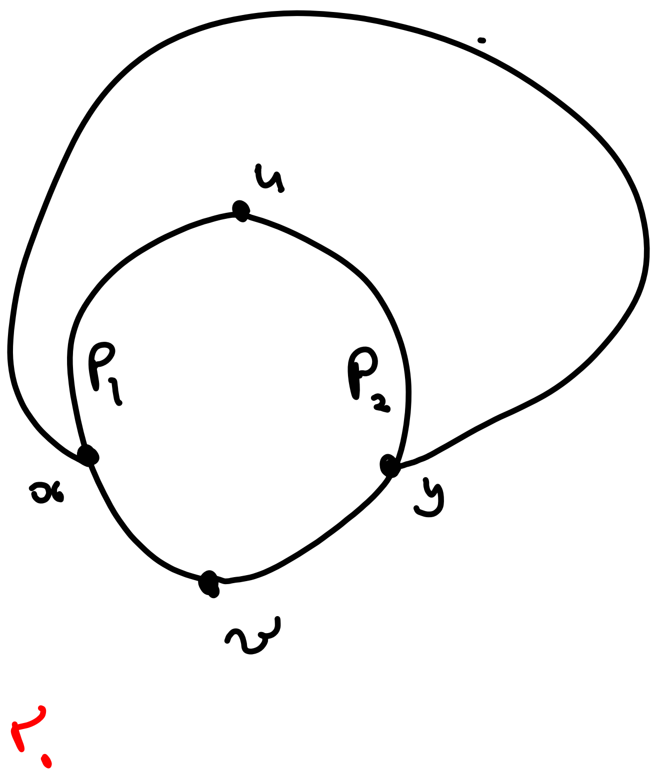
y is first intersection of Z with P_1

x is preceding intersection of Z with P_2.

Free cop occupies $P_2(x,v) + Z(x,y)$

C_1 occupies $P_1(v,y)$

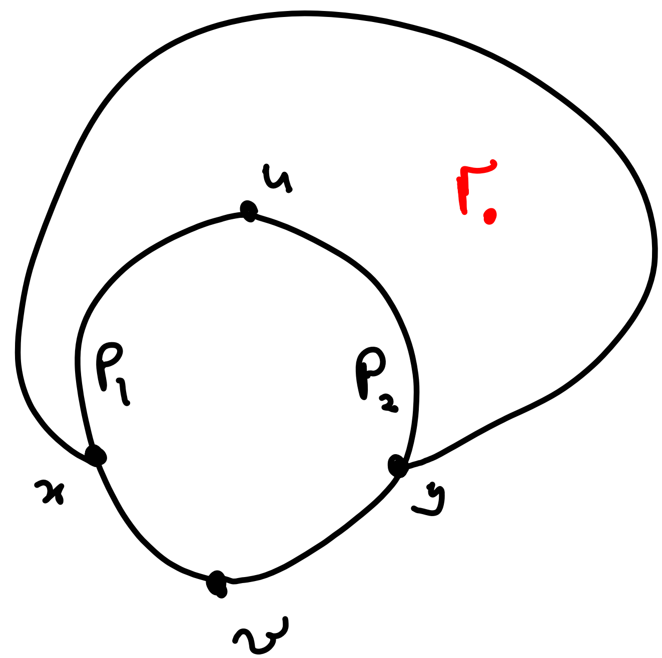




Free cop occupies

$P_2(v, y) + Z(y, x)$

C_1 occupies $P_1(x, v)$



Free cop occupies

$P_2(u, y) + Z(y, x)$

C_1 occupies $P_1(u, x)$