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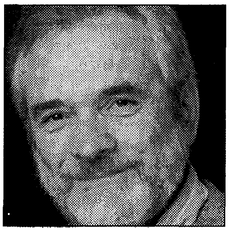

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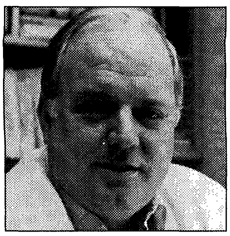

Gerald Gannon is a professor of mathematics at California State University, Fullerton, where he directs the innovative teaching option of the M.A. program. He received his doctorate in mathematics education from the University of Northern Colorado under the direction of Bob Johnson. A former basketball coach, he enjoys looking for interesting, nonroutine problems that can motivate students at different levels.

In his interview with Tom Banchoff in *Math Horizons* [February 1996, p. 19], Donald Albers quotes the famous problem of finding the different coin (lighter or heavier) among 12 otherwise identical coins using a balance scale three times. Many of us have struggled with this problem at one time or another; Martelli's turn came during the course on logic he took from the late Roberto Magari at the University of Firenze, Italy, back in 1965. After two hours of hard work, he solved it. Then, as every mathematically inclined person would do, he asked himself how the problem could be generalized and how a solution could be obtained in the simplest possible way. Over the years he tried many different approaches but was satisfied with none. Now, at last, it appears that we have found the right path.

The question to be answered is this: *What is the largest number, m, of otherwise identical coins among which a single odd coin can be detected using a balance scale n times?* Detecting includes establishing whether the coin is lighter or heavier. While the primary goal is to determine m, in fact we shall develop an explicit recursion procedure to find the coin itself. Our approach has a significant pedagogical value; it combines two simple but powerful tools that are used extensively in discrete mathematics: mathematical induction and recursion. Moreover, it demonstrates the strategic value of the old principle *divide et impera*: to solve a difficult problem, break it into simpler problems.

First simpler problem. Step one of our strategy is to suppose that the odd coin weighs less than the others. Then the largest number of coins among which the lighter one can be detected by using the balance scale n times is $m = 3^n$. This result is easily established by induction. With $n = 1$, we can find the light coin among three simply by putting one coin on each side of the scale and keeping the third in our hand. With four (or more) coins, the problem becomes impossible with only one try, since two coins must be either held or set on both pans. Using this limitation (at most three coins with one try), the largest number of coins among which we can detect the lighter one using the scale two times ($n = 2$) is 3^2. First we place three coins on each pan and hold the remaining three. If neither pan rises, the coin is among the three in our hand. Otherwise, the pan that rises contains the lighter coin. In any case, the second try will detect it.

An obvious induction argument gives $m = 3^n$ as the largest number of coins among which the lighter one can be detected using the balance scale n times.

Second simpler problem. Let's now solve, again by induction, a slightly more general version of the previous problem. Assume that the coins are divided into two groups, red and black, and the odd coin is red if heavier and black if lighter. Again, we want to show that $m = 3^n$ is the largest number of coins among which the odd one can be detected using the balance scale n times. We will suppose there is one more red coin than there are black ones. This is the situation needed in the induction argument we will use to solve our original problem. You can easily verify that the maximum number 3^n remains unchanged for other distributions of red and black coins.

With three coins only, two red and one black, the problem is easily solved: Put one red on each pan and keep the black in your hand. Regardless of the outcome you can identify the odd coin. Next, using the balance scale two times ($n = 2$), we can find the odd coin among five red and four black, 3^2 total. First, set aside one red and two black. That leaves four red and two black. Put two red and a black on each pan. If the right pan rises, then either the coin is among the two red on the left or it is the black on the right. The method of the previous case will detect it in one more try. The other two possible outcomes of the first try are handled similarly.

With 3^3 coins, 14 red and 13 black, we set aside four red and five black. That leaves ten red and eight black. Put five red and four black on each pan. Assume the right pan goes up. Then the coin is among either the five red on the left or the four black on the right. Using the procedure for the previous case, we can detect it with two more tries. The other two possible outcomes of the first try are handled similarly.

We see how the induction goes, and how it gives us a maximum number of 3^n coins among which to detect the odd one by using the balance scale n times.

One step from the solution. We are now almost ready to answer our original question. In fact, we can answer it when a little help is given, in the form of an extra coin. We assume this *test coin* is of correct weight. Now, with one weighing we can determine if a single given coin is lighter or heavier than the test coin. With $n = 2$ weighings we can find the odd one among four coins. This is how it works. Set aside one of the coins and put two of them on the left pan and the remaining coin with the test coin on the right pan. If the right pan rises, then the counterfeit coin is either among the two on the left and is heavier (these are now the "red" coins) or it is on the right and is lighter (the "black" coin). The second try, using the strategy devised for the red–black case, will detect the odd coin and will establish whether it is lighter or heavier. The two remaining possibilities are handled analogously.

With $n = 3$ we can find the odd coin among 13. We set aside four coins, put five on the left pan, and put the remaining four with the test coin on the right. Say the right pan goes up. Then the odd coin is either among the five on the left (these are now the "red" coins) or among the four on the right (the "black" coins). With the next two tries we will detect the counterfeit coin and find whether it is lighter or heavier by using our strategy for the red–black case.

Now we can see how the induction works. For a more visual display, examine Table 1. We obtain the number to be placed in row n of the "test coin" column by adding the two numbers in row $n-1$ of the "test coin" and "red–black" columns. By induction, the number in row n is $3^{n-1} + (3^{n-1} - 1)/2 = (3^n - 1)/2$.

Table 1. Weighings Needed in Various Cases

Weighings	*Lighter*	*Red–black*	*Test coin*	*General*
1	3	3	1	$1 - 1 = 0$
2	9	9	$3 + 1 = 4$	$4 - 1 = 3$
3	3^3	3^3	$9 + 4 = 13$	$13 - 1 = 12$
4	3^4	3^4	$27 + 13 = 40$	$40 - 1 = 39$
5	3^5	3^5	$81 + 40 = 121$	$121 - 1 = 120$
⋮	⋮	⋮	⋮	⋮
n	3^n	3^n	$(3^n - 1)/2$	$(3^n - 3)/2$

The solution. The last column of the table answers our original question. Notice two facts: the element in row n of the "general" column is 1 less than the element in this row of the "test coin" column, and it is 3 times the entry in row $n-1$ of the "test coin" column. To understand why these numbers are correct, let's consider the example of $n = 4$ weighings, which will detect the odd one among 39 coins. We divide 39 by 3 to get 13. Setting aside 13 coins, we place the remaining 26 in equal numbers on the two pans of the balance. Two cases are possible. First, suppose the pans are equal in weight. Then the odd coin is among the 13 set aside. So we choose a test coin from the 26 that balanced and follow the "test coin" procedure for $n = 3$. In the second case, say the left pan goes down. The 13 coins on the left are then "red" and the 13 on the right are "black." We pick one coin from the 13 left out and add it to the "red." We now have 27 coins: 14 "red" and 13 "black." To find the coin of different weight and establish whether it is lighter or heavier, we follow the red–black procedure for $n = 3$.

In this way, by applying the principle *divide et impera*, we have found a rather simple solution to our original problem, and along the way we have learned how to solve some interesting variations of it.

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Golden Mean

Mathematics comes midway between physics and metaphysics, and is more certain than either of them.

Thomas Aquinas (1225–1274), *Exposition of the De Trinitate* 6:1