Chapter 18

The Emperor and His Money

For good ye are and bad, and like to coins,
Some true, some light, but every one of you
Stamp'd with the image of the King.


![Figure 1. The Emperor's Declaration.]

"...Emperor Nu took power by overthrowing the divisive My-Nus dynasty. The Nu régime introduced many positive reforms, and in particular abolished the old (An-Tsient) irrational currency, which had his predecessor's head on it, and introduced the Nu system. The masters of the Imperial Mint, Hi and Lo, were alternately to decide the value of each new denomination, and after each decision, sufficiently many coins of this value were to be struck. All went well until Hi ordered the striking of a coin of value one, so throwing the Workers of the Mint into unemployment. They rose in a body, and threw the unfortunate Hi from the tower at the quiet end of the capital, which has been known as the Hi Tower ever since."

*My-Nus—Some Divisive Times.*
SYLVER COINAGE

Had Hi and Lo read this book, they would have realized they were only playing a game, the game of Sylver Coinage. In this the players alternately name different numbers, but are not allowed to name any number that is a sum of previously named ones. So, if \(3\) and \(5\) have been named, for example, neither of the players is allowed to play any of the numbers

\[
3, \quad 5, \quad 6 = 3 + 3, \quad 8 = 3 + 5, \quad 9 = 3 + 3 + 3, \quad 10 = 5 + 5, \quad 11 = 3 + 3 + 5, \quad ...\]

When will this game end? If neither player has played 1, 1 will still be playable. But, of course, as soon as 1 has been played, every number

\[
1, \quad 2 = 1 + 1, \quad 3 = 1 + 1 + 1, \quad 4 = 1 + 1 + 1 + 1, \quad 5 = 1 + 1 + 1 + 1 + 1, \quad ...\]

is illegal, and so the game ends. Because the player who names 1 is declared the loser, Sylver Coinage is a misère game. (Skilful players won’t spend much time on the normal play version!)

We had better point out that because the old currency had been rather irrational (with coins of value \(\sqrt{2}, e\) and \(\pi\)) the Emperor declared that there was to be a new monetary unit, the You-Nit, and the value of each coin was to be an integral number of You-Nits. (You can see the Emperor making this declaration in Fig. 1).

And recalling how people were nonplussed by the great financial scandal of the My-Nus dynasty when they had to take away Tech Kah-Weh for issuing currency of negative value, Emperor Nu decided that each coin’s value must be a positive number of You-Nits.

HOW LONG WILL IT LAST?

It might take quite a long time. To see that it can last for a thousand moves, we need only consider the game

\[
1000, 999, 998, ... , 4, 3, 2, 1.\]

And of course a thousand can be replaced by any other number, so that the game is unbounded. Many other games have this property, for example Green Hackenbush (Chapter 2) played with an infinite snake, but are boundedly unbounded because after some fixed number of moves the end will be in sight. Thus after the first move in the Hackenbush game only a finite amount of snake is left.

But Sylver Coinage is not like that! No matter what number you choose, Hi and Lo can find a way to play that number of moves so that what’s left of the game will still be unbounded. Their first thousand moves might be

\[
2^{1000}, 2^{999}, 2^{998}, ..., 2^4, 2^3, 2^2, 2^1\]

and the rest of the game can still last as long as you like:

\[
1000001, 999999, 999997, ..., 7, 5, 3, 1.\]

In other words Sylver Coinage is unboundedly unbounded. And this isn’t all. It’s unboundedly unboundedly unbounded and unboundedly like that, and so (unboundedly) on!

Nevertheless, it can’t go on for ever; in the language of Chapter 11 it’s an ender. It is because the little theorem which proves this is due to the famous mathematician J.J. Sylvester that we have called the game Sylver Coinage.
For, at any time after the first move, let g be the greatest common divisor (g.c.d.) of the moves made. Then it’s not hard to see that only finitely many multiples of g are not expressible as sums of numbers already played. So after at most this known number of moves the g.c.d. must be reduced. Eventually we must arrive at a position with g = 1 and can bound the number of moves yet to be made. So although we may not be able to bound the game after any given number of moves, we can bound the number of moves it will take to reduce the g.c.d.

**SOME OPENINGS ARE BAD**

The proof we gave in the Extras to Chapter 2 shows that from any position in Sylver Coinage there is a winning strategy for one of the two players but because of the infinite nature of the game we cannot work through all positions and guarantee to find winning strategies when they exist. In fact we do not know of (and there may not exist) any way of working out in a finite time who wins from an arbitrarily given position. But we do know the answers for some easy positions.

If at any time you name 1, you lose by definition.

If you name 2, my reply will be 3 if it’s still available, and then all larger numbers

\[
4 = 2 + 2, \quad 5 = 2 + 3, \quad 6 = 2 + 2 + 2, \quad 7 = 2 + 3, \quad 8 = 2 + 2 + 2 + 2, \quad \ldots
\]

are excluded and you will be forced to name 1.

If you name 3, then for the same reason, 2 is a good reply.

So whoever first names any of 1, 2 and 3 will lose. In particular the first three numbers are bad opening moves. What will you reply if I open with 4? Maybe 5? If so the g.c.d. becomes 1 and there will be only finitely many numbers left. We can find out which by arranging the numbers as in Fig. 2. The circled numbers are excluded because they’re multiples of 5 and these exclude the lower numbers by adding 4’s. So only 1, 2, 3, 6, 7, 11 remain.

![Diagram of Sylver Coinage](image)

**Figure 2. What's Left After \{4, 5\}.

I won’t take 1, 2 or 3. If I say 6 or 7, you’ll say the other, since these dismiss 11 and leave only 1, 2, 3 for me. So I’ll say 11 and make you say 6 or 7 instead.

\{4, 5, 11\} is a \(P\)-position.
Here's what happens after 4 and 6.

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
4 & 5 & 6 & 7 \\
8 & 9 & 10 & 11 \\
12 & 13 & 14 & 15 \\
16 & 17 & 18 & 19 \\
\end{array}
\]

Since 5 and 7 exclude all large numbers, they kill each other. Similarly for 9 and 11, and for 13 and 15, and so on.

After \{4,6\} the pairs
\( (2,3), (5,7), (9,11), \ldots, (4k+1, 4k+3) \)
for \( k \geq 1 \), are mates.

So if you open with 4, I shall respond with 6; if you open with 6, I shall respond with 4. A few similar strategies are known.

After \{8,12\} the pairs
\( (2,3), (5,7), (9,11), \ldots, (4k+1, 4k+3) \)
and
\( (4,6), (10,14), (18,22), \ldots, (8k+2, 8k+6) \)
for \( k \geq 1 \), are mates.

There is a slightly more complicated strategy showing that another good reply to 6 is 9.

After \{6,9\} mate the pairs
\( (4,11), (5,8), (7,10) \) and \( (3k+1, 3k+2) \) for \( k \geq 4 \),
but then
after 4,11 mate 5 with 7
after 5,8 mate 4 with 7
after 7,10 mate 4 with 5. 8 with 11.
We have proved that

\[
\{2,3\} \{4,6\} \{6,9\} \{8,12\}
\]

are all \(\mathcal{P}\)-positions,

and so

\[
\{1\} \{2\} \{3\} \{4\} \{6\} \{8\} \{9\} \{12\}
\]

are all \(\mathcal{N}\)-positions.

The numbers 1, 2, 3, 4, 6, 8, 9, and 12 are the only first moves for which explicit strategies have been found. You might expect that pairs \((2,3), (4k + 1, 4k + 3), (4,6), (8k + 2, 8k + 6), (8,12), (16k + 4, 16k + 12)\) provide a strategy after \(\{16, 24\}\) but unfortunately 12 is not a legal move from the position \(\{16, 24, 5, 7, 8\}\). On the other hand, for the strategies given above, both members of a pair are legal whenever one is. In fact 8 is a good reply to \(\{16, 24, 5, 7\}\) because it makes 16 and 24 irrelevant and we shall soon see that

\[
\{5,7,8\}
\]

is a \(\mathcal{P}\)-position.

We don’t know whether 24 is a good reply to 16, nor even whether 16 has any good reply.

**ARE ALL OPENINGS BAD?**

If on observing the fate of 1, 2 and 3 you thought maybe that all openings were bad, then probably our discussions of 4, 6, 8, 9 and 12 have tended to confirm your suspicions. In this section we’ll try to analyze 5 and 7. The discussion of possible replies is made a lot easier by the **clique technique**.

You’ve already seen some cliques: The number 1 forms a rather special clique all by itself; 2 and 3 form another because they exclude all larger numbers. In our discussion of \(\{4,5\}, 6\) and 7 formed a clique since they excluded 11. Cliques have the property that any reply to a clique member must also be a clique member and these two numbers must together exclude all numbers outside the clique.

We illustrate the clique technique by discussing \(\{6, 7\}\) (Fig. 3).

![Figure 3. The Cliques after (6, 7).](image)
As usual, we can disregard 1, 2 and 3 which form the innermost cliques in every position. Now 4 and 5 together exclude all larger numbers and so form a third clique. No matter what larger numbers have been named, 4 will answer 5 and 5 will answer 4. We can therefore afford to neglect them in discussing larger numbers.

Now we assert that 8, 9, 10 and 11 form the next clique, because 8 and 10 together exclude all but 9 and 11, and these together exclude all but 8 and 10. Even when some larger numbers have already been named, 8 will answer 10, 9 will answer 11, and vice versa, and we can dismiss all four from the subsequent discussion.

We now know that any good reply to any of the remaining numbers

\[
15 \quad 16 \quad 17 \\
22 \quad 23 \\
29
\]

must be another of these. We see that 15 answers 23 and vice versa since these leave only 16 and 17. Similarly 17 and 22 are mates. But since 16 excludes both 22 and 23, leaving only 15 and 17, it's a good move by itself. These five numbers form a clique, since 29 is always excluded.

16 is the unique good reply to \{6,7\}.

Table 6 in the Extras exhibits complete strategies in a similar way for all the positions

\[
\{(4,5), (4,7), (4,9), \}
\{(5,6), (5,7), (5,8), (5,9), \}
\{(6,7), (7,8), (7,9)\}
\]

In particular it shows that

\[
\{(4,5,11), (4,7,13), (4,9,19), \}
\{(5,6,19), (5,7,8), (5,9,31), \}
\{(6,7,16), (7,9,19), (7,9,24), \}
\]

are \(\mathcal{P}\)-positions.

We deduce that any good reply to 5 or 7 must be at least a two-digit number. The smallest two-digit number, 10, isn't a legal answer to 5; is it a good answer to 7? No!

\{7,10,12\} is a \(\mathcal{P}\)-position.
NOT ALL OPENINGS ARE BAD

R.L. Hutchings has proved that there can't be any good replies to 5 or 7! His main theorem is

If \(a\) and \(b\) are coprime \((g = 1)\) and \(\{a,b\} \neq \{2,3\}\), then \(\{a,b\}\) is an \(\mathcal{N}\)-position.

From this he deduces his \(p\)-theorem:

If \(p > 5\) is a prime number, 
\{\(p\)\} is a \(\mathcal{P}\)-position,

\(p\)-positions are \(\mathcal{P}\)-positions.

(For any legal reply produces a position with a g.c.d. of 1.) And from the \(p\)-theorem he deduces in turn his \(n\)-theorem:

If \(n\) is a composite number not of the form \(2^a3^b\), then 
\{\(n\)\} is an \(\mathcal{N}\)-position,

\(n\)-positions are \(\mathcal{N}\)-positions.

(Since \(n\) has a prime divisor \(p > 5\), which is a good reply.) Together these account for the first few missing numbers:

\{5\}, \{7\}, \{11\}, \{13\}, \{17\}, ... are \(\mathcal{P}\)-positions.
\{10\}, \{14\}, \{15\}, \{20\}, \{21\}, ... are \(\mathcal{N}\)-positions.
Our explicit strategies accounted for the eight smallest numbers $2^33^5$:

$$\{1\}, \{2\}, \{3\}, \{4\}, \{6\}, \{8\}, \{9\}, \{12\}$$
are $N$-positions.

But

$$\text{Nobody knows about}$$
$$\{16\}, \{18\}, \{24\}, \{27\}, \{32\}, \{36\}, \ldots$$

(We'd be glad to be proved wrong.)

**STRATEGY STEALING**

Hutchings proves his main theorem by a fine piece of strategy stealing. He considers the topmost number, $t$, that is not excluded by $\{a,b\}$ and proves that if $t$ is not a good reply, then some other number is!

We shall call $\{a,b\}$ an **end-position** because, as we'll see in a moment, the topmost number is excluded by every other legal move.

Now let's ask:

Is $t$ a good reply to $\{a,b\}$?

If the answer is "yes", then $\{a,b\}$ is an $N$-position.

If the answer is "no", then either the game is over or there is a good reply $s$ to $\{a, b, t\}$. But since $a$, $b$ and $s$ exclude $t$, $s$ is itself a good reply to $\{a,b\}$. We can say that the player to move from $\{a,b\}$ finds his strategy by stealing the second player's strategy, if he has one, for $\{a,b,t\}$.

In some cases, e.g. $\{5,9\}$, $t$ (here 31) is a good reply. But in others, e.g. $\{5,7\}$ (where $t=23$) it isn't. The strategy stealing argument only tells us that good moves exist, not what they are. Theft is no substitute for honest toil!

In general,

**An end position with $t > 1$**

is an $N$-position,

end-positions are $N$-positions.

But the end-position $\{2,3\}$ is *not* an $N$-position. This is because $t=1$ and the only legal move ends the whole game.

Why is $\{a,b\}$ an end-position if its g.c.d. is 1? In Fig. 5 we illustrate with $\{9,11\}$ for which the authors know no good reply.
Writing the numbers in 2 columns, as is our wont, we see that in each column the first excluded (circled) number is a multiple of b, so the last included numbers must differ by multiples of b. Now from any legal move s we can get to the last legal number in its column by adding a's and from this we can get to t by adding b's showing that s excludes t (e.g. s = 30 in Fig. 5). The argument also provides a proof of Sylvester's well-known formula.

\[ t = (a - 1)b - a = ab - (a + b). \]

**QUIET ENDS**

Suppose Hi and Lo have named two coprime numbers a and b and Hi is considering making the move s. Then we know that the topmost number t will be obtainable using sufficiently many coins of values s, a and b. But our argument proved that only one copy of the new coin will be needed:

\[ t = s + ma + nb. \]

More generally from a position \( \{a,b,c,\ldots\} \) we shall say that \( s \) quietly excludes \( t \) if \( t \) can be made up using any numbers of \( a,b,c,\ldots \) together with just one copy of \( s \):

\[ t = ma + nb + \ldots + s. \]

A quiet end-position is one in which the topmost legal move is quietly excluded by every number not already excluded.
If \( a \) is coprime with each of \( b \) and \( b_1 \), then
\[
S = \{a, bc, bd, be, \ldots\}
\]
is a quiet end-position if and only if
\[
S_1 = \{a, b_1 c, b_1 d, b_1 e, \ldots\}
\]
is.

**THE QUIET END THEOREM**

Thus
\[
\{7, 1 \times 3, 1 \times 4\},
\]
which is really the same position as \( \{3, 4\} \), is a quiet end-position, so that
\[
\{7, 9, 12\} = \{7, 3 \times 3, 3 \times 4\}
\]
and
\[
\{7, 15, 20\} = \{7, 5 \times 3, 5 \times 4\}
\]
are. In particular, these are end-positions and so are \( \mathcal{K} \)-positions by the strategy stealing argument. As usual we aren't told what the good replies are.

We shall use \( \{7, 9, 12\} \) and \( \{7, 15, 20\} \) to illustrate our proof of the quiet end theorem. Once again we write out the numbers in \( a \) (here 7) columns and circle the first excluded number in every column (Fig. 6). We assert that these numbers for the positions \( S \) and \( S_1 \) are in the proportion \( b : b_1 \) (3 : 5 in the example; see Fig. 7).

(a) \( S = \{7, 9, 12\} \).

(b) \( S_1 = \{7, 15, 20\} \).

*Figure 6. Circled Numbers in Proportion.*
We first see that the circled numbers for \( S \) really are multiples of \( b \). Recall that we circle \( n \) for \( S \) if \( n \) is excluded by \( S \), but \( n - a \) is not. Since \( n \) is excluded, it has the form
\[
n = ak + bm
\]
where \( m \) would be excluded by \( \{c,d,e,\ldots\} \). But if \( k \) were positive,
\[
n - a = a(k - 1) + bm
\]
would also be excluded by \( S \), so \( k = 0 \) and we have simply
\[
n = bm.
\]
Now our assertion is that \( bm \) is circled for \( S \) only if \( b,m \) is circled for \( S_1 \). Now \( b,m \) is certainly excluded by \( S_1 \) and so is circled unless
\[
b,m - a
\]
is also excluded. But then we must have
\[
b,m - a = ak + b,m'
\]
for some \( m' \) excluded by \( \{c,d,e,\ldots\} \), and
\[
b,m = a(k + 1) + b,m',
\]
showing that \( b \) divides \( k + 1 \) since it is coprime with \( a \). We can now divide by \( b \), and multiply by \( b \) to obtain
\[
bm = ak' + bm'
\]
for some positive number \( k' \), showing that
\[
bm - a = a(k' - 1) + bm'
\]
was excluded, and \( bm \) was not circled for \( S \).

In its modest way, the quiet end theorem is quite powerful. It often gives the quietus to infinitely many replies with a single blow.

No odd number is a good reply to \( \{16,24\} \).
For 1 clearly isn’t and if a is any other odd number then \{a,2,3\} is really the same as the quiet end position \{2,3\}. By the quiet end theorem \{a,16,24\} is a quiet end-position and so an \(N\)-position.

In a similar way it proves that \{4,6\} and \{6,9\} are \(P\)-positions without bothering to provide a detailed strategy. Let’s use it to discuss the position \{8,10\}. After \{4,5\} we found that the only remaining moves were

\[
1, \quad 2, \quad 3, \quad 6, \quad 7, \quad 11,
\]

so after \{8,10\} the only remaining even numbers will be twice these,

\[
2, \quad 4, \quad 6, \quad 12, \quad 14, \quad 22.
\]

The quiet end theorem enables us to say that any good reply to \{8,10\} must be in one of these two sets, for otherwise it is an odd number \(a\) excluded by \{4,5\} so \(a,4,5\) and therefore \(a,8,10\) will be quiet end-positions. Now,

- 1 loses instantly,
- (2,3) are mated as usual,
- (4,6) eliminate 8,10 and will mate, as will
- (7,11) (see \{6,7\} in Table 6 in the Extras) and
- (12,14) by our strategy for \{8,12\}.

So 22 is the only hope for a good reply to \{8,10\}. We shall see later that

\[
\{8,10,22\} \text{ is a } P\text{-position.}
\]

**DOUBLING AND TRIPLING?**

Note that the \(P\)-position \{8,10,22\} is the *double* of \{4,5,11\}. Our \{8,12\} strategy shows that all \(P\)-positions arising in the \{4,6\} strategy have doubles that are also \(P\)-positions. Maybe every \(P\)-position doubles to another? No! For \{5,6,19\} is \(P\), but \{10,12,38\} is answered by 7 since \{10,12,38,7\} is really the same as \{7,10,12\}.

Maybe the *triple* of every \(P\)-position is another? No! This time \{4,5,11\} is \(P\), but \{12,15,33\} is answered by 5 since \{5,12,33\} is a \(P\)-position, as we’ll soon see.

**HALVING AND THIRDING?**

Nevertheless there are many \(P\)-positions whose doubles and triples are still \(P\). We conjecture:

\[
\text{If } \{2a,2b,2c,\ldots\} \text{ is } P
\]
\[
\text{so is } \{a,b,c,\ldots\}?
\]

\[
\text{and}
\]
\[
\text{If } \{3a,3b,3c,\ldots\} \text{ is } P
\]
\[
\text{so is } \{a,b,c,\ldots\}?
\]
FINDING THE RIGHT COMBINATIONS

How should you start a game of Sylvester Coinage? Now that you know so much you will perhaps name 5 for your first move. You now have a strategy for every move I might make and probably feel a little safe. But those stolen strategies are firmly locked inside that little safe you're feeling and more than sensitive fingers are needed to find the right combinations.

You know the first few: 1 needs no reply and you should make the pairs (2,3), (4,11), (6,19), (7,8) and (9,31). Is there any general rule? In trying to answer this question for you we went to a lot of trouble and eventually found a fairly efficient way of breaking open the safe. But the winning combinations it reveals (Fig. 8) suggest that there is no simple answer.

Figure 8. The Stolen Secrets of Safe Number 5.

Let's take a closer look at a position in which 5 and some other numbers have been named. If we were to write the numbers in five columns as usual we would circle 0 and just four other numbers $a, b, c, d$ in the 1-, 2-, 3-, 4-columns respectively, as in Fig. 9. We now make a three-dimensional table of $P$-positions using just three of these numbers as headings and the fourth as an entry.
Table 1(a) shows the case in which \( a \) is the entry and \( b,c,d \) the row, column and layer headings. Tables 1(b,c,d) have \( b,c,d \) as entries.

Table 1(a) Entries \( a \) for \( \mathcal{A} \)-positions \( \{5,a,b,c,d\} \).

Table 1(b). Entries \( b \) for \( \mathcal{A} \)-positions \( \{5,a,b,c,d\} \).
Some positions will appear repeatedly because a heading is redundant. These are indicated by bold figures. For example

\{5,6,12,13,14\}, \{5,6,17,13,14\}, \{5,6,22,13,14\}, ...

are really the same position because 12 = 6 + 6 is redundant and so we have a column of 6's in layer 14 of Table 1(a). In \{5,6,12,18,19\} both 12 and 18 are redundant, so the 19 layer of that table is almost entirely made up of 6's.
In \{5,16,7,13,9\} it is the entry \(16 = 7 + 9\) that is redundant, so 16 can be replaced by any of 21, 26, 31, 36, 41, ...

and we have written \(16+\) to indicate this. Really an entry \(n+\) is short for infinitely many entries
\[n, n+5, n+10, n+15, n+20, \ldots\]

The entries in Table I(a) were computed in lexicographic order, by making due allowance for these repetitions and otherwise entering the least number \(5k+1\) not appearing earlier in the same row, column or file.

You'll probably find the method easier to follow in Table 2 which deals with positions \(\{4,a,b,c\}\) in a similar way. This time each entry is the smallest number \(b = 4k+2\) which has not appeared earlier in its row or column and an entry \(b+\), shorthand for
\[b, b+4, b+8, b+12, b+16, \ldots\]
is made when \(b=2a\) or \(2c\). It can be deduced from the quiet end theorem that other kinds of repetition will not appear.

Table 3 gives pairs \(x<y\) for which \(\{4,x,y\}\) is already a \(\mathcal{P}\)-position, extracted from an extended version of Table 2, kindly calculated for us by Richard Gerritse. It seems that the ratio \(y/x\) approaches \(2.56\ldots\).

As soon as 4 or 5 arises in your game you should refer to the appropriate one of these tables. If 6 turns up first, see the corresponding Table 7 in the Extras.
Table 2. Values of $b$ for which \{4, a, b, c\} is a $\mathcal{B}$-position.
Table 3. Pairs \( x,y \) for which \( \{x,y\} \) is a \( \varphi \)-position.

WHAT SHALL I DO WHEN \( g \) IS TWO?

Example: \( \{8,10,22\} \)

Apparently we have to examine infinitely many possible replies. Fortunately there is a way of doing this in a finite time. A similar method will work for any position with \( g = 2 \).

Let’s see how the position will look after some play from \( \{8,10,22\} \). If 1, 2 or 3 has been played, we know what to do. Otherwise the only even numbers that can have been played are

\[
4, 6, 12, 14
\]
and the even part of the position must look like one of

\[
\{4,6\} \{4,10\} \{6,8,10\} \\
\{8,10,12,14\} \{8,10,12\} \{8,10,14\} \{8,10,22\}
\]

What odd numbers have been played? If the least of them is \(n\), then since \(\{8,10,22\}\) excludes all of

16, 18, 20, 22, 24, ...

we can suppose that the only relevant odd numbers are among

\[n, n+2, n+4, n+6, n+12, n+14.\]

And if any even moves have been made they will restrict the possibilities still further. For instance if 6 has been played we can suppose that the odd numbers are one of the sets

\[n, n+2, n+4 \quad n, n+2 \quad n, n+4 \quad n\]

Table 4 shows the status of the positions classified in this way. Since the last four columns repeat indefinitely, this finite table contains the information for every odd \(n\). How was it computed and why is it periodic?

Let's take a typical entry:

\[\{8,10,14,n,n+6,n+12\}.\]

From this position there are three kinds of option:

(a) the even numbers 4, 6 or 12,
(b) the small odd numbers \(m \leq n-14\),
(c) the large odd numbers \(n-12, n-10, n-8, n-6, n-4, n-2, n+2, n+4\).

Case (a) leads to a position (in an earlier segment of the table) with even part

\[\{4,10\}, \{6,8,10\} \text{ or } \{8,10,12,14\}\]

and we can suppose that these have already been analyzed and found to be ultimately periodic in \(n\).

A case (b) move leads to

\[\{8,10,14,m\}\]

since \(m\) excludes \(n\) and all larger odd numbers. If there is any odd \(m\) for which this is a \(\mathcal{P}\)-position, then \(\{8, 10, 14, n, n+6, n+12\}\) will be an \(\mathcal{K}\)-position for all \(n \geq m+14\). If not, we can reject moves in case (b).

Finally, case (c) moves either leave \(n\) unchanged or decrease it by at most 12. We conclude that the outcome of every position in the table is computed in a fixed way from

ultimately periodic information (case (a)),
ultimately constant information (case (b)), and
information in the last few columns (case (c));

it must therefore be ultimately periodic in \(n\).
<table>
<thead>
<tr>
<th>( (4,6) ) ( {n, n+2} )</th>
<th>( n )</th>
<th>( \frac{n}{n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (4,10) ) ( {n, n+2} )</td>
<td>( \frac{n}{n} )</td>
<td>( \frac{n}{n} )</td>
</tr>
<tr>
<td>( (6,8,10) ) ( {n, n+2, n+4} )</td>
<td>( \frac{n}{n} )</td>
<td>( \frac{n}{n} )</td>
</tr>
<tr>
<td>( (8,10,12,14) ) ( {n, n+2, n+4, n+6} )</td>
<td>( \frac{n}{n} )</td>
<td>( \frac{n}{n} )</td>
</tr>
<tr>
<td>( (8,10,12) ) ( {n, n+2, n+4, n+6, n+2, n+4, n+6} )</td>
<td>( \frac{n}{n} )</td>
<td>( \frac{n}{n} )</td>
</tr>
<tr>
<td>( (8,10,14) ) ( {n, n+2, n+4, n+6, n+2, n+4, n+6, n+4, n+6, n+4, n+6} )</td>
<td>( \frac{n}{n} )</td>
<td>( \frac{n}{n} )</td>
</tr>
<tr>
<td>( (8,10,22) ) ( {n, n+2, n+4, n+6, n+2, n+4, n+6, n+4, n+6, n+4, n+6, n+2, n+4, n+6, n+2, n+4, n+6, n+4, n+6, n+4, n+6} )</td>
<td>( \frac{n}{n} )</td>
<td>( \frac{n}{n} )</td>
</tr>
</tbody>
</table>

Table 4. The Position \((8,10,22)\).
Every position with \( g = 2 \) can be handled in this way. When we have computed enough to verify the period, we can decide in particular whether there is any good reply. For \{8,10,22\} there isn't one, so it is a \( \mathcal{P} \)-position.

**THE GREAT UNKNOWN**

We can best describe our knowledge in terms of the number \( g \). When

\[
q = 1
\]

the position is bounded so you can find what to do by working through all positions. Of course this might take a long time even if one of our theorems already tells you the outcome. We know that there must be a good reply to \{31,37\} but don't know any method which guarantees to find one in the next millenium. When

\[
q = 2
\]

the method we have just described will compute the outcome in a finite but probably even longer time. If

\[
g \text{ is divisible by a prime } p > 5
\]

then \( p \) is a good reply when it hasn't already been named, when of course there isn't any.

The authors have only been able to examine a few particular positions with other values of \( g \). Table 8 in the Extras contains a complete discussion of \{6,9\}. Although this is a two-dimensional table, a periodicity develops which enables us to analyze the position to infinity. Maybe a similar thing happens for some other positions with \( g = 3 \). We computed a much larger three-dimensional table for \{8,12\} (\( g = 4 \)), but could detect no structure outside the range covered by our explicit strategy.

16 is the first opening move whose status is in doubt. We don't know whether \{16\} has a good reply nor even any way of finding out in any finite time. You might consider working upwards testing each possible reply in turn and hoping to detect some structure, but even this is impossible. We don't know any way to test the reply 24, say, in any finite time. We don't even know how to test 100, say, as a possible reply to \{16,24\}!

The quiet end theorem often eliminates infinitely many replies, for example all odd replies to \{16\} or to \{16,24\}, but it never eliminates any reply that would be infinitely hard to analyze.
Table 5. Status of Subsets of \((6,7,8,9,10,11,12)\) and Known Good Replies.

Even members of set at left; Odd members at head. Bracket is closed when all good replies are known, so that \([\cdot]\) indicates a \(p\)-position, but \(\cdot\) always indicates an \(\mathcal{M}\)-position for which no good reply is known. The last entry contains all primes greater than 3; and may contain some entries \(2^3\).

<table>
<thead>
<tr>
<th>{6,8,10}</th>
<th>{7,9}</th>
<th>{7,11}</th>
<th>{7}</th>
<th>{9,11}</th>
<th>{9}</th>
<th>{11}</th>
<th>{}</th>
</tr>
</thead>
<tbody>
<tr>
<td>[8]</td>
<td>[15]</td>
<td>[8,9]</td>
<td>[8,9]</td>
<td>[4]</td>
<td>[9,10,11]</td>
<td>[5,7]</td>
<td>[7,10]</td>
</tr>
<tr>
<td>[6]</td>
<td>[8,10,11]</td>
<td>[8,9]</td>
<td>[16]</td>
<td>[4,7]</td>
<td>[8,10,11]</td>
<td>[4]</td>
<td>[8,13]</td>
</tr>
<tr>
<td>{8,10,12}</td>
<td>[4,5,6]</td>
<td>[4]</td>
<td>[13]</td>
<td>[4]</td>
<td>[13,14,15]</td>
<td>[23]</td>
<td>[6]</td>
</tr>
<tr>
<td>{8,12}</td>
<td>[5]</td>
<td>[6]</td>
<td>[6]</td>
<td>[5]</td>
<td>[11,13,14]</td>
<td>[9,13]</td>
<td>[14]</td>
</tr>
<tr>
<td>{10,12}</td>
<td>[4]</td>
<td>[4,6]</td>
<td>[6]</td>
<td>[5]</td>
<td>[11,13,14]</td>
<td>[9,13]</td>
<td>[14]</td>
</tr>
<tr>
<td>{12}</td>
<td>[6]</td>
<td>[15]</td>
<td>[27]</td>
<td>[10]</td>
<td>[8,10,11]</td>
<td>[23]</td>
<td>[6]</td>
</tr>
<tr>
<td>{8,10}</td>
<td>[4,5,6]</td>
<td>[4]</td>
<td>[5,6,11]</td>
<td>[23]</td>
<td>[13,15]</td>
<td>[6,7]</td>
<td>[22]</td>
</tr>
<tr>
<td>{8}</td>
<td>[5]</td>
<td>[6]</td>
<td>[6,10]</td>
<td>[5]</td>
<td>[12]</td>
<td>[21]</td>
<td>[12]</td>
</tr>
<tr>
<td>{10}</td>
<td>[4]</td>
<td>[4,6]</td>
<td>[8]</td>
<td>[12]</td>
<td>[12]</td>
<td>[5]</td>
<td>[5]</td>
</tr>
<tr>
<td>{}</td>
<td>[6]</td>
<td>[19,24]</td>
<td>[24,34]</td>
<td>[6]</td>
<td>[5,7,11,13,\ldots]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5 tells you the outcome and all the good replies we know to every position made from the numbers

\[6, 7, 8, 9, 10, 11, 12\]

(if 4 or 5 is involved, Tables 2, 3, 1 and Fig. 8 go much further). If you can add any more to this table or decide whether any number \(2^3\) is a good opening move we would like to hear from you.

ARE OUTCOMES COMPUTABLE?

We can prove that there must be a way of programming a computer to find the outcome of \(\{n\}\) even though we don't know what that way is! The reason is:

There can only be finitely many good opening moves \(2^3\).

For no one of these can divide any other, so that no two can have the same value of \(a\) or the same value of \(b\). So if \(2^a3^b\) is such a number with \(a_0\) as small as possible, and \(2^c3^d\) is any other, then we must have \(b < b_0\) and so there are at most \(b_0 + 1\) such numbers, say \(n_1, n_2, \ldots, n_k\). We suspect there are none!

If you only knew what these numbers were, then you could program your machine with PORN (Fig. 10) and work out the outcome of any \(\{n\}\). This argument shows that in the purely technical sense this is a computable function of \(n\), even though we don't know what function it is.
THE ETIQUETTE OF SILVER COINAGE

START

IS \( n \) A PRIME \( \geq 5 \)?
YES NO

PRINT "P" AND STOP

IS \( n \) DIVISIBLE BY SUCH A PRIME?
YES NO

PRINT "N" AND STOP

IS \( n = n_1 \),
OR \( n = n_2 \),
OR \( n = n_3 \)?
YES NO

PRINT "P" AND STOP
PRINT "N" AND STOP

Figure 10. PORN, A Program Which Decides if \( \{n\} \) is \( \mathcal{P} \) or \( \mathcal{N} \).

THE ETIQUETTE OF SILVER COINAGE

Few Western readers can understand the subtleties of etiquette in the oriental country from which our game comes. But at least we can save you from the more obvious gaffes by pointing out that in Sylver Coinage it is customary for a player who knows he is winning to resign by naming 1, 2 or 3. This quaint custom is said to originate in the tradition that Hi, who could see much further than Lo, nobly took upon himself the fate that was about to befall his beloved brother.

When it's plain to all the world that you have a win, any move but 1 will insult your opponent, but in other cases we advise you to name 3 (2 is possible, but may be misunderstood). If your opponent concurs in your analysis, he will respond with 2, but you have allowed him to express another opinion by naming 1. (Replies to 3 other than 1 or 2 may also be available but their nuances are harder to interpret.)

Of course, one of the greatest insults you can offer is to name 1, 2 or 3 at the very start of the game, for this is the philosopher Hu Tchings' prerogative, at least until someone finds a new way to win.
EXTRAS

CHOMP

Here is a game with similar rules to Sylver Coinage. For some fixed number $N$, the players
alternately name divisors of $N$ which may not be multiples of previously named numbers.
Whoever names 1 loses. If $N = 432 = 2^4 \cdot 3^3$, for example, a move is essentially to eat a square
(e.g. 36) from the chocolate bar in Fig. 11, together with all squares below and/or to the right of
it. Square number 1 is poisoned!

<table>
<thead>
<tr>
<th>10</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>12</td>
<td>24</td>
<td>48</td>
</tr>
<tr>
<td>9</td>
<td>18</td>
<td>36</td>
<td>72</td>
<td>144</td>
</tr>
<tr>
<td>27</td>
<td>54</td>
<td>108</td>
<td>216</td>
<td>432</td>
</tr>
</tbody>
</table>

**Figure 11.** Chomping at a Chocolate Bar.

The first few $\mathcal{P}$-positions are shown in Fig. 12. Strategy stealing shows that rectangles larger
than $1 \times 1$ are $\mathcal{N}$-positions; the replies are unique if either side is at most 3, but Ken Thompson
found that $4 \times 5$ and $5 \times 2$ bites both answer $8 \times 10$.

The arithmetic form of the game is due to Fred. Schuh, the geometric one to David Gale.

ZIG-ZAG

Two players alternately name distinct numbers (which are allowed to be fractional or negative)
and the game ends as soon as the resulting sequence contains either an increasing subsequence
(zig) of length $a$ or a decreasing one (zag) of length $b$. The normal play $a + 1$, $b + 1$ game is really
the same as the misère $a$, $b$ game, and so we consider only the latter.

Zig-Zag, which was suggested to us by S. Fajtlowicz, sounds difficult to analyze, but fortunately
there is a rather clever transformation into a geometrical game like Chomp. We regard square
$(r, s)$ in Fig. 13 as eaten if the number sequence so far contains a rising zig of length $r$ and a sagging
zag of length $s$ that end with the same number. Then the moves are as in Chomp except that the
first move may eat square $(1, 1)$ only, and the innermost square eaten on any subsequent move
must be adjacent to a previously eaten square. The squares $(a, 1)$ and $(1, b)$ are poisoned, so play
really goes on inside the outlined $a - 1$ by $b - 1$ chocolate bar of Fig. 13.
Good replies are shown linked.

Values of \( \alpha \) for which \( (x, \alpha) \) is a \( \mathcal{P} \)-position.

Winning bids from \( a \times b \) rectangles.

**Figure 12.** \( \mathcal{P} \)-positions in Chomp.
If $a \geq 3$, $b \geq 3$ and $a + b \leq 17$, the first player wins the misère $a$-Zig, $b$-Zag game, because David Seal's calculations show that the corresponding $a - 1$ by $b - 1$ chocolate bars are $\mathcal{N}$-positions.

By assigning Heads to Horizontal edges and Tails to Vertical ones we get an equivalent game with coin sequences, involving moves of a head rightwards over tails or a tail leftwards over heads, and Seal used this idea to compute Fig. 14 showing all $\mathcal{P}$-positions for which the uneaten part of the chocolate bar fits inside a $5 \times 5$ square.

To find $\mathcal{P}$-positions in both Chomp and Zig-Zag, we used the tabular technique of Chapter 15, and the Clique Technique of this one.
Heavy edges must be part of the original boundary of the chocolate bar; other edges may be.

and reflections about the diagonal through the poisoned square

\( f(x,y) = f(y,x) \).  \( f(0,0) = 1. \)
If \( x < 2^\circ y \), \( f(x,y) = f(x,y - 2^\circ) \).
If \( 2^\circ x, y < 2^\circ \), \( f(x,y) = 0 \),
except that \( f(2,2) = -2^\circ \).

"Concave symmetric" positions are \( R \) positions if they fit inside a \( 7 \times 7 \)
square or have \( 3 \) diagonal squares.

Figure 14. \( R \)-positions for Zig-Zag.
MORE CLIQUES FOR SYLVER COINAGE

To follow the cliques in Table 6, we advise you to set out the remaining numbers as we did in Figs. 2, 3, 4 for the cases {4,5}, {6,7} and {7,10,12}. Numbers not mentioned are excluded by a good reply.

<table>
<thead>
<tr>
<th>position</th>
<th>replies</th>
<th>strategy, with cliques indicated by ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>{4,5}</td>
<td>11!</td>
<td>1'?[2,3] ][6,7] ]11!</td>
</tr>
<tr>
<td>{4,7}</td>
<td>13!</td>
<td>1'?[2,3] ][5,6] ][9,10] ]13!</td>
</tr>
<tr>
<td>{5,7}</td>
<td>8!</td>
<td>1'?[2,3] ][4,6][9,13] ][11,13] ]8!</td>
</tr>
<tr>
<td>{5,8}</td>
<td>7!</td>
<td>1'?[2,3] ][4,11] ][6,9] ]7!</td>
</tr>
<tr>
<td>{5,9}</td>
<td>31!</td>
<td>1'?[2,3] ][4,11][6,8][7,13] ][12,16] ][17,21] ][22,26] ][31!</td>
</tr>
<tr>
<td>{6,7}</td>
<td>16!</td>
<td>1'?[2,3] ][4,5] ][8,9][8,10][8,11][9,10][9,11] ][15,23][17,22] ][16!</td>
</tr>
<tr>
<td>{7,8}</td>
<td>5!</td>
<td>1'?[2,3] ][4,13][6,9][6,10][6,11] ]5!</td>
</tr>
<tr>
<td>{7,9}</td>
<td>19!24!</td>
<td>1'?[2,3] ][4,10][5,13][6,8][6,10][6,11] ][12,15][17,20][22,26][29,33] ][19!24!</td>
</tr>
<tr>
<td></td>
<td></td>
<td>after [7,9,22,26][12,17][19,24][15,20]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[7,9,29,33][12,15][17,20][19,31][22,26][24,26] ...</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[7,9,19] [12,15][15,17][15,20][22,24][29,31]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[7,9,24] [12,15][17,20][19,22][26,29]</td>
</tr>
</tbody>
</table>

Table 6. Some Complete Strategies for Sylver Coinage.

5-PAIRS

The safe combinations {5,x,y} are of three types. In the top drawer in Fig. 8 are those with y so much larger than x that the coordinates {a,b,c,d} are {x,2x,3x,y}. For these it seems that y/x tends to 3. But are there infinitely many numbers in the top drawer?

The middle drawer contains the remaining ones for which x+y is a multiple of 5. It seems that for these, x and y always differ by 1 or 2.

In the bottom drawer we have arranged the pairs with coordinates {x, y, x+y, 2y} where x and y are in the order given. It seems that here, as in the second drawer, y/x tends to 1.

POSITIONS CONTAINING 6

As in our other analyses, we write the numbers in six columns and circle 0 and five other numbers a,b,c,d,e, one in each of the 1-, 2-, 3-, 4- and 5-columns respectively. We tabulate $\Phi$-positions by entries c in a 4-dimensional table (Table 7) whose coordinates, a,b,c,d,e are congruent to 1,2,4,5, modulo 6.

Entries outside the areas enclosed by full lines are found by repeating entries according to the arrows, where appropriate. The tables for $b=8$, $d=4$ and $b=8$, $d=16$ can be extended indefinitely by repeating the portions between the pecked lines and increasing all entries by 12 or 60 respectively. The two tables with $d=10$, $b=8$ or 14 contain no further entries. All further entries in that for $b=14$, $d=16$ are 15.
Entries are $e = 3$, modulo 6. Bold type indicates that the row or column heading is redundant. Entries repeat indefinitely according to the arrows.

Table 7. $\mathcal{P}$-Positions Containing 6,
Table 8. A Complete Discussion of \{(6,9)\}.

The table represents a complete discussion of positions containing \{(6,9)\}. Reduced positions contain one, or possibly two neighboring, numbers of form $3k + 1$, and of form $3k + 2$; pairs of rows and columns refer to the latter and former possibilities. Positions represented by cells outside the crenellated line are not reduced. The pattern within the rectangular quadrant continues indefinitely. The minus signs denote $\mathcal{N}$-positions, the plus signs $\mathcal{P}$-positions and the “=” signs $\mathcal{N}$-positions which, at a casual glance at the pattern, might be mistaken for $\mathcal{P}$-positions.
SYLVER COINAGE HAS INFINITE NIM-VALUES

If we make naming 1 an *illegal* rather than a *stupid* move, Sylver Coinage becomes a normal play rather than a misère play game and we could consider adding it to other games using the Sprague-Grundy theory. However since some positions have infinitely many options, we can expect infinite nim-values and indeed they happen!

For example, $\mathcal{G}(2,2n+3)=n \ (n \geq 0)$, so $\mathcal{G}(2)=\omega$. On the other hand, $\mathcal{G}(3,3n+1,3n+2)=1 \ (n \geq 1)$, so $\mathcal{G}(3)=1$. Here are some other nim-values:

\[
\begin{align*}
 k &= 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 13 \ 14 \ 15 \ 16 \ 17 \ 19 \ 20 \ 22 \ 23 \ 25 \ 26 \ 28 \ 29 \ 31 \ 32 \\
\mathcal{G}(3,k) &= 2 \ 3 \ 1 \ 4 \ 6 \ 1 \ 7 \ 8 \ 9 \ 11 \ 1 \ 12 \ 14 \ 15 \ 17 \ 19 \ 20 \ 21 \ 23 \ 24 \ 26 \ 27 \\
\mathcal{G}(4,k) &= ? \ 3 \ 0 \ 5 \ ? \ 8 \ 1 \ 9 \ 14 \ 4 \ 15 \ ? \ 16 \ 19 \ ? \\
\mathcal{G}(4,6,4n-1,4n+1) &= 1 \\
\mathcal{G}(4,6,4n+1,4n+3) &= 0 \ (n \geq 1) \\
\mathcal{G}(5,6) &= 7, \mathcal{G}(5,7) = 8, \mathcal{G}(5,8) = 10, \mathcal{G}(6,7) = 9, \mathcal{G}(3,3n-1,3n+1) = 5 \ (n \geq 6), \\
\mathcal{G}(3,9n-8,9n-4) = \mathcal{G}(3,9n+2,9n+7) = \mathcal{G}(3,9n+8,9n+13) = 10 \ (n \geq 3).
\end{align*}
\]

A FEW FINAL QUESTIONS

Is there any effective technique for computing the outcome and all good replies for the *general position*?

If the game is played "between intelligent players", is the first person to make the game bounded the loser?

Is there a winning strategy of bounded length?

Is there an $N$-position with $g>1$ for which all good replies lead to positions with $g=1$?

Is $\mathcal{G}(4)=\omega+1$ or is $\mathcal{G}(4)=6$, say?

REFERENCES AND FURTHER READING


Martin Gardner, Mathematical Games, Sci. Amer. 228 1, 2, 5 (Jan. Feb. May 1973)


