Frieze Patterns, Triangulated Polygons and Dichromatic Symmetry

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The material in this article is not new, except perhaps for the final formulae in Section 3, but we have tried to present proofs of some known results in an intuitive and recreational manner.

1 Frieze Patterns

A frieze, in architecture, is "the part of the entablature between the architrave and cornice, often ornamented with figures" [14]; the Elgin Marbles, which formed the frieze on the Parthenon, are a well known example. But a frieze is also "a decorated band along the top of a room wall" [14], and in mathematics the term "frieze pattern" is reserved for any plane pattern that repeats regularly in one direction; such a pattern has translational symmetry, and a simple example is shown in the top row of Figure 1, where the basic translation is indicated by an arrow.

Frieze patterns can have other types of symmetry in addition, and they can be classified according to their symmetry types. The seven different symmetry-types are all shown in Figure 1, labelled according to one of the standard notations [16, p. 683; 13] in which the symbols $I$, $m$, 2, and $g$ stand for translation, mirror, rotation and glide-reflexion. Type $Im$ has a horizontal line of symmetry or mirror line. Type $mI$ has vertical mirror lines; these occur at two different places in the pattern. Type $mm$ has both horizontal and vertical mirror lines.

Type $I2$ has no mirror lines, but it has centres of 2-fold rotational symmetry, which we shall call simply centres of symmetry, indicated by dots; in type $I2$ these centres of symmetry occur at two different places in the pattern. Type $mg$ has vertical mirror lines and centres of symmetry.

In type $Ig$ there is clearly some type of symmetry connecting the motifs in the top and bottom halves of the pattern, but the pattern has no mirror lines or centres of symmetry. We can transform a motif at the top to an adjacent motif at the bottom by means of a translation and a horizontal reflexion performed simultaneously. This type of transformation is called a glide reflexion (the relation between successive footprints along a snowy path), and we can indicate it by two half-arrows.

Any translation, reflexion, rotation or glide reflexion that transforms a pattern into itself is called a symmetry of the pattern.

Glide reflexions occur in Types $Im$, $mm$, and $mg$ also, but in these types the glide reflexion is the product of two "more basic" symmetries of the pattern, either a translation and a horizontal reflexion, or a rotation and a vertical reflexion; this is not the case in Type $Ig$. 

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Type \( mm \) has centres of symmetry, at the intersection of each vertical mirror line with the horizontal mirror line. Rotation through 180° about such a centre, which transforms the whole pattern to itself, can be achieved as the product of the reflexions in the two mirror lines through the centre.

We shall not attempt to show here that these are the only types of symmetry that a frieze pattern can have. It should be emphasized that symmetry types are not concerned with the style of a pattern; Figure 2 shows another pattern, from Turkey, whose symmetry-type is \( ml \), having only vertical mirror lines.

Figure 2: A frieze pattern from Turkey.
2 Triangulated Polygons and Frieze Patterns

The first author once asked an audience of a hundred students of mathematics to look at the pattern in Table 1 and find the simple rule connecting each number with its neighbors and allowing the pattern to be extended indefinitely to the right and left. After an embarrassingly long time it was necessary to break the suspense by explaining that any four numbers forming a diamond, such as

\[
\begin{array}{cc}
  b \\
  a & d \\
  c
\end{array}
\]

satisfy the relation \(ad - bc = 1\), which may also be written \(c = (ad - 1)/b\); this is called the unimodular rule. Later, to test the effect of a brilliant brain, the same pattern was shown to Paul Erdős; he needed only a few seconds!

Using the same unimodular rule to construct other similar patterns with initial and final rows of zeros, it is soon observed that the diagonal sequence

\[
0 \quad 1 \quad 1 \quad 1 \quad 2 \quad 1 \quad 0
\]

starting at the top left corner of Table 1 (from which the entire pattern can be constructed) can be replaced by any sequence

\[
0 \quad 1 \quad a \quad c \quad ... \quad 1 \quad 0,
\]

where \(a, c, \ldots\) are positive integers each dividing the sum of its two neighbors; for instance

\[
0 \quad 1 \quad 2 \quad 3 \quad 7 \quad 4 \quad 1 \quad 0.
\]

All such patterns turn out to be frieze patterns with symmetry type \(Ig\), or in some special cases \(Im\), \(mg\), or \(mm\) as we shall see later. This remarkable periodicity is easy to prove when the number of rows is sufficiently small [3, pp. 90, 175]; for greater numbers of rows the proof is more tricky, and the search for it caused many restless nights [5; 6].

Instead of beginning with a diagonal sequence, we could just as well begin by writing down a “suitable” periodic sequence below the first horizontal row of 1s; but what type of sequence is a “suitable” sequence to provide eventually another row of 1s? The surprisingly simple answer was supplied by J. H. Conway [3, pp. 87–88, 180].

A convex \(n\)-gon needs \(n-3\) diagonals to triangulate it, that is, to divide it up into \(n-2\) triangles. For instance, Figure 3 [2, p. 174] shows one of the four essentially different triangulations of a heptagon [16, p. 355; 15]. At each vertex of the heptagon is written the number of triangles that come together at that vertex, and these numbers, repeated in their cyclic order, form the third row of numbers in Table 1. Conway found that a sequence of integers is “suitable” for forming a frieze
pattern, using the unimodular rule, if and only if it arises in this way from a triangulated polygon. In this manner a triangulated $n$-gon yields a sequence of period $n$ (or possibly a divisor of $n$) and a frieze pattern of $n + 1$ rows (including the top and bottom rows of 0s).

![Figure 3: A triangulation of a heptagon.](image1)

![Figure 4: Triangle 125 adjoined to side 14 of Figure 3.](image2)

If we remove the top triangle from the triangulated polygon in Figure 4, we obtain Figure 3 (actually, a distortion of Figure 3, but the exact shape of the polygons is not important), so Figure 4 is obtained by adjoining a triangle to Figure 3. Table 2 shows how the frieze pattern associated with Figure 4 is obtained from Table 1: the triangular portions in Table 1 have been separated to leave diagonal channels between them, and each new number in the channels is the sum of its two nearest neighbors in the separated portions. This observation leads to a proof by induction of Conway's result that every triangulated polygon has an associated frieze pattern of positive integers with glide-reflexion symmetry. We shall sketch the proof.

![Table 2.](table_image)

Suppose, as an inductive hypothesis, we have proved that every triangulated $n$-gon has an associated frieze pattern of $n + 1$ rows, bordered at the top and bottom by rows of 1s and 0s, with the "vertex numbers" of the polygon in the third and antepenultimate rows, satisfying the unimodular rule, and with glide-reflexion symmetry. Any given triangulated $(n + 1)$-gon may be obtained by adjoining a triangle to a suitable $n$-gon. This $n$-gon has an associated frieze pattern with a glide reflexion, which can be divided into triangular portions as in Table 1, taking into account the position where the extra triangle is to be adjoined to the polygon. Separate the triangular portions as...
in Table 2, and insert a new positive integer at each position in the channels by adding together the two nearest positive integers in the triangles. The new frieze pattern of \( n \) rows of positive integers has the vertex numbers of the triangulated \((n + 1)\)-gon in its third and antepenultimate rows, and it has glide-reflexion symmetry. Also it satisfies the unimodular law, because if
\[
\begin{array}{cc}
b & a \\
c & d
\end{array}
\]
is a unimodular diamond, then so are the two diamonds contained in
\[
\begin{array}{ccc}
b & b+d & a+c \\
a+c & d & c
\end{array}
\]
since
\[
a(b + d) - b(a + c) = (a + c)d - (b + d)c = ad - bc = 1.\]

Hence the new frieze pattern is the frieze pattern associated with the \((n + 1)\)-gon. The induction can be started, since the unique triangulated 4-gon certainly has an associated frieze pattern; hence by induction every triangulated polygon has an associated frieze pattern with symmetry type \(lg\).

A school teacher might well show a class of young children how to triangulate a convex polygon and how to construct the corresponding frieze pattern. This is a nice exercise in multiplication and division, because any mistake is liable to cause its own penalty in the form of a fraction or a negative number, whereas accuracy yields the pleasant surprise of finding that the process ends with another row of 1s followed by a row of 0s.

A triangulated polygon with 2-fold rotational symmetry, as in Figure 5a, gives a frieze pattern of type \(1m\). A triangulated polygon with a line of symmetry, as in Figure 5b, gives a frieze pattern of type \(mg\), and one with two perpendicular lines of symmetry (which must have rotational symmetry also), as in Figure 5c, gives a frieze pattern of type \(mm\).

![Figure 5: Symmetrical triangulations.](image)

In Section 3 we shall show that every multiplicative pattern (i.e., every pattern satisfying the unimodular rule) with a finite number of rows and bounded by rows of 0s and 1s, and with no other 0s in it, is a frieze pattern with glide-reflexion symmetry. The fact that every such frieze pattern whose entries are positive integers is associated with a triangulated polygon (the converse of what we have just proved) is verified by simply reversing the steps in the above proof; but before we can
take the general step that is equivalent to reducing Table 2 to Table 1, we have to show that every frieze pattern of positive integers contains at least one 1 in the third row. The details are given in [3].

In [3] there is a formula for the numbers in the third row of a frieze pattern in terms of the numbers in any diagonal. In Section 3 we shall give a formula for all the numbers in a frieze pattern in terms of the numbers in any diagonal.

Figure 6: The seven different diagonals of Table 1.

Conway [3, pp. 93, 183] discovered also that the triangulated polygon provides, in a simple way, not only the third row in the frieze pattern but also the diagonal sequences in the pattern. For this purpose new numbers are assigned to the vertices, as in Figure 6, which shows the same triangulation as in Figure 3 but with different labels. Each vertex in turn is labelled 0. The number 1 is assigned to each of the two or more vertices that are joined to the vertex marked 0. Then whenever a triangle has two marked vertices, the third vertex takes their sum. The labels thus accumulated all round the polygon provide one of the diagonal sequences in the frieze pattern.

One surprising corollary of this numbering scheme is the following statement [3, pp. 93, 183]:

*Every frieze pattern of integers either contains a 4 or consists entirely of Fibonacci numbers!*

The above definition of a frieze pattern of numbers may be modified in various significant ways. Duane Brolin [1] considered modifying the unimodular rule \(ac - bd = 1\), but he had to allow the symmetry to be reduced from \(I_2\) to \(I_1\). Shephard's "additive" frieze patterns [17] will be discussed in Section 5.

In connexion with determinants and continued fractions [6, pp. 306–308], regular polytopes [7, pp. 22, 54–57, 141–147, 165–178] and "polygonometry" [4, pp. 204–205; 10], it is natural to abandon the restriction to integers; but then, of course, there is no longer a connexion with triangulated polygons.
3 More About Multiplicative Frieze Patterns

If a multiplicative pattern contains 0s other than those in the first and last rows, it need not repeat regularly; an example is shown in Table 3, where $a$, $b$, $c$ etc. can be any numbers.

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 \\
a & 0 & c & 0 \\
-1 & -1 & -1 & -1 & -1 \\
0 & b & 0 & d \\
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

Table 3.

We shall henceforth assume without explicit mention that all numbers in our patterns are nonzero, except in the first and last rows.

If we consider the portion of a multiplicative pattern shown below

\[
\begin{array}{c}
a_1 \\
a_2 & b_1 \\
a_3 & b_2 \\
b_3 \\
\end{array}
\]

we easily deduce from the unimodular law that

\[
(a_1 + a_3)/a_2 = (b_1 + b_3)/b_2.
\]

Consider now the multiplicative pattern in Table 4.

\[
\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
A & B & C \\
x & y & a_1 \\
z & a_2 & b_1 \\
a_3 & b_2 & \cdots \\
b_3 & \cdots & \cdots \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Table 4.

By applying (1) to consecutive triples in the table, we find that $(A + 0)/1 = (0 + A')/1$, i.e. $A = A'$. Similarly $A' = A''$. Hence the third row (and therefore all the other rows also) repeats regularly, so we have a frieze pattern, and because $A = A'$ we have glide-reflexion symmetry (since the pattern can equally well be built up from the bottom).
Table 5 now shows a typical frieze pattern; the entries in the second row are all equal to 1, and 0, $f_0$, $f_1$, $f_2$, . . . is one of the diagonals. Applying (1) to consecutive triples along the diagonal indicated in the table, we find that

$$(f_3 + f_5)/f_4 = (0 + e_{35})/e_{34} = e_{35}$$

($= a_4$ in the notation of [3]). In the same way we obtain the general formula

$$a_s = e_{s-1,s+1} = (f_{s-1} + f_{s+1})/f_s,$$

which gives us values for all the entries in the third row except for $e_{06}$.

Write $u_i = 1/f_i$; then the general entry in Table 5 is given by the formula

$$e_{ij} = f_i f_j (u_i u_{i+1} + u_{i+1} u_{i+2} + \ldots + u_{j-1} u_j) \quad (i < j).$$

To verify this, we simply have to check that with these values for the $e_{ij}$ the unimodular law is satisfied throughout the table; this is left to the reader.

We now have a value for the “missing entry” $e_{06}$ in the third row, which was not given by the previous formula.

4 Black-and-White Symmetry

Suppose that we color each motif in a frieze pattern either black or white (and imagine the “background” to have some third neutral color) in such a way that each symmetry of the original pattern either maps black to black and white to white or else interchanges black and white; we then have a perfect black-and-white coloring (or dichromatic coloring) of the frieze pattern. As an example, consider Figure 7, where our original pattern of type mg has been colored in three different ways. The first coloring has reflexions in vertical mirror lines that do not interchange the colors, indicated by unbroken lines, and rotations about centres of symmetry that do interchange the colors, indicated by white dots. The second has vertical mirror lines that do interchange the colors, indicated by broken lines, and centres of symmetry that do not, indicated by black dots. In the third coloring, both the mirror lines and the centres of symmetry interchange the colors.
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If in the first coloring in Figure 7 we look at the black motifs only, we see that these form a pattern of type \( ml \); hence this coloring is labelled \( mg/ml \). For similar reasons the other two colorings are labelled \( mg/12 \) and \( mg/1g \). This notation is an adaptation of the one devised by Schwarzenberger for colored plane symmetry patterns and described in [9].

Two patterns have the same type of black-and-white symmetry if they have the same arrangement of broken and unbroken mirror lines or arrows, and white and black dots. For example, Figure 8 shows two different colorings of our pattern of type \( mm \), but the two colorings are of the same type because they have the same arrangement of vertical mirror lines, alternately broken and unbroken, and both have a broken horizontal mirror line.

There are seventeen types of black-and-white symmetry [8; 17], but Figure 9 shows only those that will interest us here, namely those possessing a glide reflection that interchanges the colors.
Figure 9: Symmetries where a glide reflection swaps the colors.
5 Additive Frieze Patterns

In [18] Shephard considers additive frieze patterns, in which the rule for “completing the diamond” is

\[ a + d = b + c + 1 \]

and the first and last rows consist of 0s; one such pattern is shown in Table 6.

\[
\begin{align*}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 4 & 1 & 5 & 3 & 1 & 1 & 4 & 1 \\
3 & 1 & 4 & 4 & 5 & 7 & 3 & 1 & 4 & 4 & 5 \\
2 & 3 & 3 & 7 & 6 & 6 & 2 & 3 & 3 & 7 \\
4 & 3 & 1 & 5 & 7 & 4 & 4 & 3 & 1 & 5 & 7 \\
4 & 0 & 2 & 4 & 4 & 1 & 4 & 0 & 2 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{align*}
\]

Table 6.

Table 7 shows the constant additive frieze pattern with seven rows; it is a simple exercise in induction to show that a constant additive frieze pattern with \( n + 2 \) rows must have \( n/2 \) as the constant number in the second row.

\[
\begin{align*}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\frac{1}{2} & 2\frac{1}{2} & 2\frac{1}{2} & 2\frac{1}{2} & 2\frac{1}{2} & 2\frac{1}{2} & 2\frac{1}{2} \\
4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4\frac{1}{2} & 4\frac{1}{2} & 4\frac{1}{2} & 4\frac{1}{2} & 4\frac{1}{2} & 4\frac{1}{2} & 4\frac{1}{2} \\
4 & 4 & 4 & 4 & 4 & 4 & 4 \\
2\frac{1}{2} & 2\frac{1}{2} & 2\frac{1}{2} & 2\frac{1}{2} & 2\frac{1}{2} & 2\frac{1}{2} & 2\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{align*}
\]

Table 7.

If we subtract Table 7 from Table 6, and then multiply all the resulting numbers by 2 to avoid fractions, we obtain Table 8.

\[
\begin{align*}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & -3 & 3 & -3 & 5 & 1 & -3 \\
-2 & -6 & 0 & 0 & 2 & 6 & -2 \\
-5 & -6 & 0 & 0 & 2 & 6 & -2 \\
0 & -2 & -3 & 5 & 3 & 3 & -3 \\
3 & -5 & 0 & 0 & 2 & 6 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{align*}
\]

Table 8.

Each diamond in this table satisfies the rule

\[ a + d = b + c; \]

we shall call frieze patterns satisfying this rule simple frieze patterns, because the rule is so simple. We see that the frieze pattern in Table 8 possesses a glide reflection that multiplies each number by
-1, thus transforming positive to negative and vice versa. This corresponds to the black-and-white symmetry type \(lg/11\).

It is easy to prove that a simple frieze pattern always has this type of positive-negative symmetry. Taking a pattern with six rows as an example, suppose that one diagonal consists of the entries \(0\ a\ b\ c\ d\ 0\), which we shall write in the form

\[
\begin{array}{cccccc}
0 & 0 & a & 0 & b & 0 & c & 0 & d & 0 & 0.
\end{array}
\]

Using the "simple diamond rule" we quickly verify that the complete pattern is

\[
\begin{array}{cccccccc}
0 & 0 & a-a & b-b & c-c & d-d & 0 & 0 \\
& a-0 & b-a & c-b & d-c & 0 & d & 0 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
a-d & b-0 & c-a & d-b & 0 & c & a-d & 0 \\
b-d & c-0 & d-a & 0 & b & a-c & 0 \\
b-c & e-d & d-0 & 0 & a & b & c & 0 \\
c-c & d-d & 0 & 0 & a-a & b-b & 0 & 0
\end{array}
\]

Table 9.

which clearly shows the sign-changing glide reflexion. This result is implicit in [18] but is expressed differently there.

There is a one-to-one correspondence between additive frieze patterns and simple frieze patterns (although we obscured this by multiplying by 2 to obtain Table 8); hence any additive pattern with a finite number of rows must repeat and is therefore a frieze pattern, and it possesses a hidden type of symmetry, which is displayed clearly in the corresponding simple frieze pattern as a positive-negative glide reflexion.

It is an interesting exercise to construct simple frieze patterns with extra positive-negative symmetries, corresponding to all the black-and-white symmetry types shown in Figure 9. Table 10 shows one whose symmetry type is \(mg/ml\); the reader may like to construct others, some of which must perforce contain a row of 0s in the middle.

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 5 & -5 & 3 & -3 & -3 & 3 & -5 \\
0 & 10 & 0 & -2 & 0 & -6 & 0 & -2 \\
5 & 5 & 3 & -5 & -3 & -3 & -5 & 3 \\
8 & 0 & 8 & 0 & -8 & 0 & -8 & 0 \\
3 & 3 & 5 & -3 & -5 & -5 & -3 & 5 \\
0 & 6 & 0 & 2 & 0 & -10 & 0 & 2 \\
3 & 3 & -3 & 5 & -5 & -5 & 5 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

Table 10.


References


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