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A GAME OF COPS AND ROBBERS

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Let G be a finite connected graph. Two players, called cop C and robber R, play a game on G according to the following rules. First C then R occupy some vertex of G. After that they move alternately along edges of G. The cop C wins if he succeeds in putting himself on top of the robber R, otherwise R wins. We review an algorithmic characterization and structural description due to Nowakowski and Winkler. Then we consider the general situation where n cops chase the robber. It is shown that there are graphs on which arbitrarily many cops are needed to catch the robber. In contrast to this result, we prove that for planar graphs 3 cops always suffice to win.

1. Introduction

Let G be a finite connected undirected graph. Two players called *cop* C and *robber* R, play a game on G according to the following rules: First C and then R occupy some vertex of G. After that they move alternatively along edges of G. The cop C wins if he succeeds in putting himself on top of R (if he 'catches' R). If the robber R can prevent C from ever catching him, then R wins. It is obvious that for every graph G one of the players must win, in fact, if C has a winning strategy, then he should succeed in catching R after at most $n(n-1)+1$ moves (n = number of vertices in G) since he can avoid repeated positions.

The game in the form as just described, i.e. with complete information on both sides, has also been studied by Nowakowski and Winkler [5], Quilliot [8] and possibly others, see also Smith [9]. There are of course, a multitude of interesting variations. One could for example, allow complete information only when C and R are at most a distance d apart (C and R have 'eye-contact'). With no information at all, we enter the topic of search games, see e.g. Gal [2] and Parsons [6, 7].

Let us denote by $\mathscr{C}$ the class of cop-win graphs (i.e. those graphs on which C has a winning strategy) and by $\mathscr{R}$ the class of robber-win graphs. For all terms which are not explained in the text, see Harary [3]. All graphs considered are assumed to be finite, undirected and connected.

Examples. (a) An obvious family of cop-win graphs are the *trees*. The vertex occupied by C partitions the tree into 2 components and each time C moves along the unique path toward R, the robber-component is reduced by at least one vertex. By starting at the center of the tree, C clearly minimizes the number of his moves.

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(b) A similarly obvious family of robber-win graphs are the *cycles* of length at least 4 since the robber R can always position himself at distance 2 from C.

(c) The same argument can be used to show that, more generally, every *regular non-complete graph* is in $\mathscr{R}$.

One important distinction has to be made: In the *active* version of the game the robber *must* move whenever it is his turn, in the *passive* version he may also stay put if he so chooses. The cop must move in either case. We may, of course, subsume the passive under the active version by adding a loop to every vertex. Trees, cycles and regular graphs are cop-win resp. robber-win under both versions. But, in general, a graph may well change its character. The example in Fig. 1 is the smallest graph where C wins in the active version (by starting at the top vertex) but loses in the passive version.

In this paper we concentrate solely on the more natural passive version.

In Sections 2 and 3 we review an algorithm and a structural result concerning $\mathscr{C}$ found by Nowakowski and Winkler [5]. In Section 4 we consider the general situation where k cops go after the robber. This section includes what we consider the prettiest result found so far, namely that in any planar graph 3 cops always suffice to catch a robber. Let other people draw the necessary conclusion from this result.

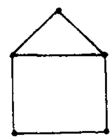

Fig. 1.

2. An algorithmic characterization of $\mathscr{C}$

Suppose G is a cop-win graph. Let us take a look at the situation just before the robber's last move. Since R may sit still and C is supposed to catch him with his next move, R and C must be joined by an edge. Since R cannot evade C, all neighbors of R must also be neighbors of C, $N(R) \subseteq N(C)$. Let us call a pair (p, d) of vertices a *pitfall* together with its *dominating vertex* if $N(p) \cup \{p\} \subseteq N(d)$. Hence if G has no pitfalls, then G is necessarily in $\mathscr{R}$.

Example. The Octahedron of Fig. 2 is in $\mathscr{R}$ since it has no pitfalls.

Lemma 1. *Let p be a pitfall of G and $\tilde{G} = G - p$ the graph with p and all incident edges deleted. Then $G \in \mathscr{C}$ iff $\tilde{G} \in \mathscr{C}$.*

Proof. Let d be a dominating vertex to p and suppose $\tilde{G} \in \mathscr{C}$. C can then extend his winning strategy in $\tilde{G}$ to all of G by pretending R is on d whenever R enters p and

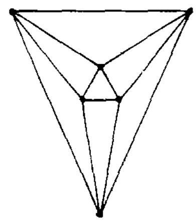

Fig. 2.

moving according to his strategy in $\tilde{G}$. If, on the other hand $\tilde{G} \in \mathscr{R}$, then R can extend his strategy by the same identification of p with d. □

Theorem 1. *G is a cop-win graph iff by successively removing pitfalls (in any order) G can be reduced to a single vertex.*

Proof. The lemma says that the win-character of a graph is not changed by removing pitfalls. Hence we end up either with a graph with at least two vertices and no pitfalls (in which case G is in $\mathscr{R}$ by the remark before the example) or with a single vertex (in which case G is in $\mathscr{C}$). □

Example. In Fig. 3 the circled vertices are pitfalls at every stage. By the theorem the graph is cop-win.

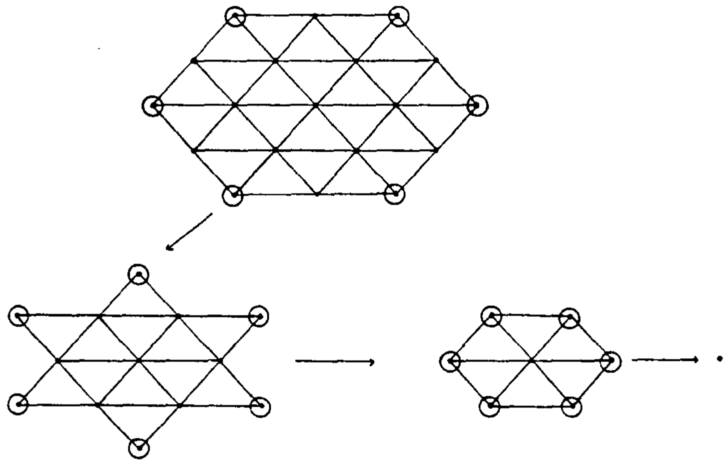

Fig. 3.

Remark. As the search for pitfalls requires only comparison of neighbor lists, the algorithm is clearly polynomial in the number of vertices.

3. $\mathscr{C}$ is a variety

Let G and G' be graphs with vertex-sets V and V' and edge-sets E and E'. By the *product* $G \times G'$ we mean the graph with vertex-set $V \times V'$ where $(v, v'), (w, w')$ is an edge iff $v = w$, $\{v', w'\} \in E'$ or $\{v, w\} \in E$, $v' = w'$ or $\{v, w\} \in E$, $\{v', w'\} \in E'$.

Lemma 2. *If $G, G' \in \mathscr{C}$, then $G \times G' \in \mathscr{C}$.*

Proof. Since the cop has a winning strategy in each of the graphs he may apply them simultaneously to track R down in $G \times G'$. □

Let G be a graph and H a subgraph. By a *retraction* from G to H we mean a map φ: $V(G) \to V(H)$ which is the identity on $V(H)$ and for which $\{v, w\} \in E(G)$ implies $\{\varphi(v), \varphi(w)\} \in E(H)$. H is called a *retract* of G if there is a retraction from G to H.

Lemma 3. *If $G \in \mathscr{C}$ and H is a retract of G, then $H \in \mathscr{C}$.*

Proof. Let us turn the statement around: $H \in \mathscr{R} \Rightarrow G \in \mathscr{R}$. Let φ: $V(G) \to V(H)$ be a retraction map. We know R has a winning strategy on H. The following strategy extends this to all of G: R stays in H and pretends the whole game is taking place in H by identifying $v \equiv \varphi(v)$ for all $v \in V(G)$. That is, if C moves from v to w, R pretends C really moved from $\varphi(v)$ to $\varphi(w)$ (the edge $\{\varphi(v), \varphi(w)\}$ exists by the definition of a retraction) and makes his move according to his strategy in H. It is easily seen that this works. (In fact, we have used the same argument in Lemma 1, since φ: $V(G) \to V(\tilde{G})$ with $\varphi(p) = d$, $\varphi(v) = v$, $v \neq p$, is a retraction.) □

A class of graphs which is closed under the operations finite direct product and retraction is called a *variety* of graphs. Nowakowski and Rival [4] have recently demonstrated the usefulness of this structural concept, see also Duffus and Rival [1]. By the lemmas we infer:

Theorem 2. *The class of cop-win graphs is a variety.*

It may be a worthwhile (but probably very hard) problem to determine the irreducible elements in this variety in order to complement the algorithmic characterization of $\mathscr{C}$ by a purely structural description.

4. More cops to come

As we have seen in Section 1, regular graphs allow the robber a very simple evading strategy as do other classes of graphs which contain cycles of length ≥ 4. Let us then give the cop player C a better chance by allowing him, say, k cops

$C_1, \ldots, C_k$. At every turn C may move any subset of $[C_1, \ldots, C_k\}$ but, of course, at least one.

Definition. For a graph G, $c(G)$ denotes the minimal number of cops needed for the player C to win. $c(G)$ is called the *cop-number*.

$c(G)$ is obviously bounded above by the vertex covering number of G, since by placing his cops at a minimal cover C catches R with his next move.

We now prove three results, one favoring the robber, the other two favoring the cop.

Theorem 3. *Let G be a graph with minimum degree $\delta(G) \geq n$ which contains no 3- or 4-cycles. Then $c(G) \geq n$.*

Proof. Let C have $n-1$ cops at his disposal. We show first that the covering number $\alpha_0(G) \geq n$. Let $v_1, \ldots, v_{n-1}$ be any $n-1$ vertices of G and $w \notin \{v_1, \ldots, v_{n-1}\}$. (Such a w exists since $\delta(G) \geq n$.) Suppose the neighborhood of w is $N(w) = \{v_1, \ldots, v_k, w_1, \ldots, w_{l-k}\}$ with $w_i \notin \{v_1, \ldots, v_{n-1}\}$. Then $l \geq n$, $k \leq n-1$ and thus $l-k \geq 1$. As there are no 3- or 4-cycles we infer $N(w_i) \cap N(w_j) = \{w\}$ for $i \neq j$. If $\{v_1, \ldots, v_{n-1}\}$ were a point-cover for G, the $N(w_i)$'s would have to contain at least one v_j, $j \geq k+1$, accounting together with $v_1, \ldots, v_k$ for at least $l \geq n$ vertices v_i, a contradiction. Hence after C makes his opening move, say $\{c_1, \ldots, c_{n-1}\}$, R is able to place himself on a vertex r which is not equal to and not adjacent to any of the c_i's. But R can keep up this situation because, after every move of C, at most $n-1$ of R's neighbors are occupied by cops or immediately adjacent to them (no 3- or 4-cycles!), thus allowing R to go to the free neighbor. □

Examples. The Petersen graph P and the Dodecahedron D depicted in Fig. 4 satisfy the conditions of the theorem with $n = 3$. Hence $c(P) \geq 3$ and $c(D) \geq 3$. That, in fact, $c(P) = c(D) = 3$ is easily seen and will also follow from Theorems 5 and 6.

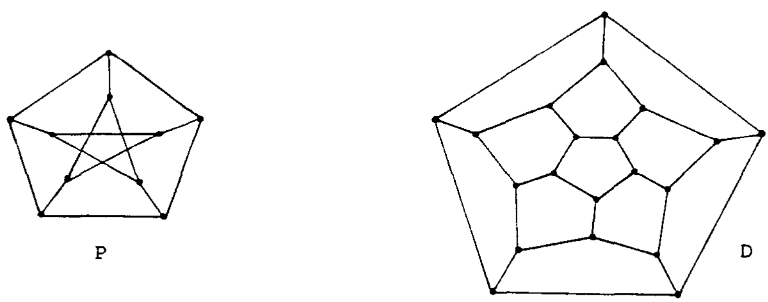

Fig. 4.

The question naturally arises whether, for any n, there are graphs which will satisfy the conditions of Theorem 3.

Theorem 4. *To every $n \in \mathbb{N}$ there exists an n-regular graph without 3- or 4-cycles. Hence, for every n, there exists a graph G with $c(G) \geq n$.*

Proof. For $n=1$ and $n=2$, K_2 and the 5-cycle C_5 will do. Also, C_5 is 3-colorable. Assume, inductively, that we have constructed an n-regular 3-colorable graph G without 3- or 4-cycles. Let G_1, G_2, G_3 be 3 copies of G and color all four graphs isomorphically with 3 colors. We construct a new $n+1$-regular graph according to Fig. 5. The figure means: If, e.g., a vertex in G_1 is colored by 3, then we join it with the corresponding (isomorphic) vertex in G_2. After all vertices have been joined (to exactly one other vertex) we interchange the colors 3 and 1 in G_1, 2 and 1 in G_2 and 3 and 2 in G_3. The resulting graph now satisfies all requirements. □

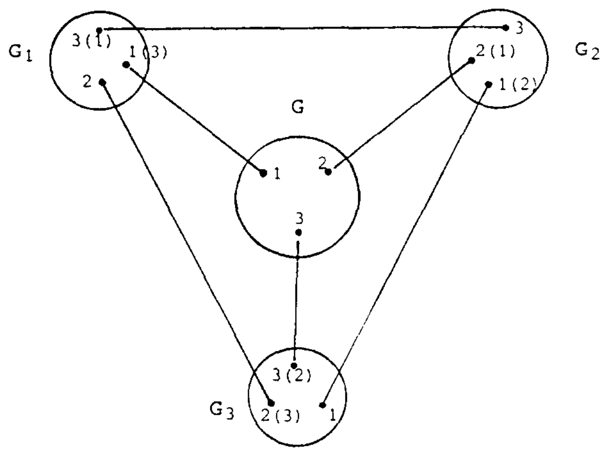

Fig. 5.

There is a complementary result to Theorem 3 bounding the cop-number from above. We state it only for degree ≤ 3. It can be generalized to arbitrary degree, but the conditions become unwieldy.

Theorem 5. *Let G have maximal degree $\Delta(G) \leq 3$ and suppose any two adjacent edges are contained in a cycle of length at most 5. Then $c(G) \leq 3$.*

Proof. Suppose after C's move the cops C_1, C_2, C_3 occupy the vertices c_1, c_2, c_3 and the robber vertex r. We choose 3 paths from the c_i's to r which use all incident edges of r (such paths exist by the condition of the theorem) and among all such triples of paths we select a triple P_1, P_2, P_3 whose total length $l = l_1 + l_2 + l_3$, $l_i = l(P_i)$ for

$i=1,2,3$, is minimal. (*Note:* P_1, P_2, P_3 need not be disjoint.) We now show that, whatever R does, C has an answering move $\{c_1, c_2, c_3\} \to \{c_1', c_2', c_3'\}$ with $l' < l$. Thus after a finite number of moves we must have $l < 3$ which means that R has been caught. Suppose r is adjacent to three vertices a_1, a_2, a_3 with $P_1 = \{c_1, \dots, a_1, r\}$, $P_2 = \{c_2, \dots, a_2, r\}$, $P_3 = \{c_3, \dots, a_3, r\}$. (The cases when $\deg(r) \le 2$ are dealt with by analogous arguments.) If R keeps still, then each cop moves on his path toward R, and we have $l' \le l - 3 < l$. Suppose now R moves to a_1. If $l_1 = 1$ and, in particular, if $\deg(a_1) = 1$, then C_1 sits on a_1, and we are finished. If $\deg(a_1) = 2$ and $l_1 \ge 2$, then again all cops move on their paths toward r and we have

$$l' \le (l_1 - 2) + l_2 + l_3 = l - 2 < l.$$

If $\deg(a_1) = 3$ and $l_1 \ge 2$, let u be the vertex adjacent to a_1 which is not on P_1. By hypothesis, the path r, a_1, u is contained in a cycle of length ≤ 5. As this cycle must use one of the edges $\{a_2, r\}$ or $\{a_3, r\}$ suppose it uses $\{a_2, r\}$ with v being the possible 5th vertex (see Fig. 6). Let all cops move toward r to c_1', c_2', c_3'. By using the paths $P_1' = \{c_1', \dots, a_1\}$, $P_2' = \{c_2', \dots, a_2, v, u, a_1\}$, $P_3' = \{c_3', \dots, a_3, r, a_1\}$ we have

$$l' \le l(P_1') + l(P_2') + l(P_3') \le (l_1 - 2) + (l_2 + 1) + l_3 = l - 1 < l. \qquad \square$$

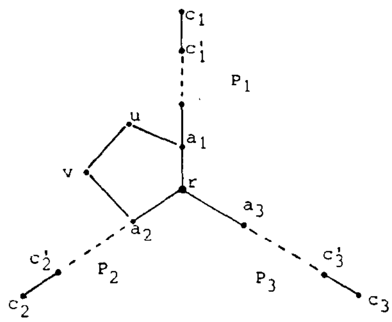

Fig. 6.

Quite possibly, $c(G) \le 3$ holds for any graph with $\Delta(G) \le 3$ and, more generally, $c(G) \le n$, but this is an open question. If so, the n-regular graphs without 3- or 4-cycles would all have cop-number n.

After we have seen in Theorem 4 that there are grahs which require arbitrarily many cops let us turn to the opposite question whether there is a sufficiently large class of graphs where the cop-number is universally bounded. Since by Euler's relation there is no planar graph G with $\delta(G) \ge 4$ and without 3- or 4-cycles we may venture the guess that $c(G) \le 3$ for any planar graph, and this is indeed so.

Theorem 6. *We have $c(G) \le 3$ for any planar graph G.*

The proof of the theorem rests on the following lemma.

Lemma 4. *Let G be any graph, $u, v \in V(G)$, $u \neq v$ and $P = \{u, v_1, \ldots, v_t = v\}$ a shortest path between u and v. We assume that at least two cops are in the play. Then a single cop C on P can, after a finite number of moves, prevent the robber R from entering P. That is, R will be immediately caught if he moves onto P.*

Proof. Denote by $d(x, y)$ the distance (=length of a shortest path) between x and y. As is well-known, d satisfies the triangle inequality. For simplicity, let us denote the path $P = \{u = 0, 1, 2, \ldots, t = v\}$. Suppose after C's move the cop is on vertex $c \in V(P)$ and the robber is on $r \in V(G)$ and assume

$$d(r, z) \geq d(c, z) \quad \text{for all } z \in V(P). \tag{$*$}$$

Claim. No matter what the robber does, the cop, by moving in the appropriate direction on P, can preserve condition $(*)$. This of course, means that the robber will be caught should he enter P.

If the robber stays put, then so does the cop. (We assume that there is at least one other cop somewhere who now makes some move). Suppose R goes from r to s, then

$$d(s, z) \geq d(r, z) - 1 \geq d(c, z) - 1 \quad \text{for all } z \in V(P).$$

If $z_0 \in V(P)$ exists with $d(s, z_0) = d(c, z_0) - 1$, then C, by moving toward z_0, also reduces the distance by 1 and $(*)$ still holds. Hence for the robber to be really threatening there must be vertices $x, y \in V(P)$ with, say, $x < c < y$ and $d(s, x) = d(c, x) - 1$, $d(s, y) \leq d(c, y)$ or $d(s, x) \leq d(c, x)$, $d(s, y) = d(c, y) - 1$ (see Fig. 7). This is, however, impossible since by the triangle inequality and the minimality of P,

$$d(x, y) \leq d(s, x) + d(s, y) \leq d(c, x) + d(c, y) - 1 = d(x, y) - 1,$$

contradiction. It remains to be shown after a finite number of moves the cop C can force condition $(*)$. First, C moves to some $c \in V(P)$. By the same argument as before $d(r, z) < d(c, z)$ can only hold for z's on P on one side of c. By moving in the direction of z, $(*)$ is clearly eventually forced. □

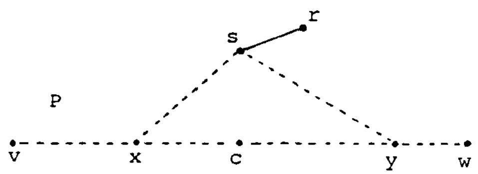

Fig. 7.

Notice that a cop who controls a path P in the sense of the lemma also controls every subpath of P.

Proof of the Theorem. Let G be embedded in the plane. The idea of the proof is to assign at each stage i to R a certain subgraph R_i, the *robber territory,* which contains all vertices which R may still safely enter, and to show that, after a finite number of cop-moves, R_i is reduced to $R_{i+1} \subsetneq R_i$. Hence, eventually, there is no vertex left for the robber to go.

At the start the 3 cops C_1, C_2, C_3 occupy some vertex e_0. After R makes his move, say to r_0, the robber territory R_0 is defined to be the graph component of $G - e_0$ which contains r_0. Suppose inductively that at stage i (after R's move) one of the two following situations arise:

(a) Some cop C is on vertex u, R is on r, and R_i is the component of $G-u$ containing r (Fig. 8). Note that it is also the opening situation.

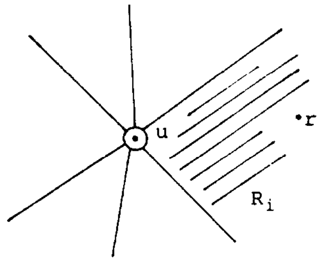

Fig. 8.

(b) P_1 and P_2 are two u,v-paths of length ≥ 1, disjoint except for u and v. By the planarity of G, $P_1 \cup P_2$ partitions G into $P_1 \cup P_2$, an interior and an exterior region. Suppose without loss of generality P occupies some vertex r in the exterior region E. P_1 is a shortest u,v-path in $P_1 \cup P_2 \cup E$, P_2 is a shortest u,v-path in $P_1 \cup P_2 \cup E$ among all such paths which are disjoint from P_1. Cop C_1 on $c_1 \in V(P_1)$ controls P_1 in the sense of the lemma, and cop C_2 on $c_2 \in V(P_2)$ controls P_2. The robber-territory is defined to be $R_i = E$ (Fig. 9).

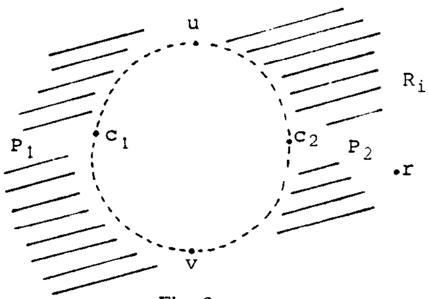

Fig. 9.

Case (a). Suppose u has only one neighbor v in R_i. Let C move to v. If $r = v$, then the game is over. Otherwise, no matter what R does the component R_{i+1} of $G - v$ containing the robber is contained in $R_i - v$. Hence we are back to case (a) with $R_{i+1} \subsetneq R_i$. Suppose now u has at least two neighbors a and b in R_i and let P be a shortest path in R_i between a and b. By the lemma, one of the two free cops, say D, controls P after a finite number of moves and we arrive at case (b) with $P_1 = \{a, u, b\}$, $P_2 = P$ (or $P_1 = P$, $P_2 = \{a, u, b\}$ if $\{a, b\} \in E(G)$) and $R_{i+1} \subseteq R_i - V(P) \subsetneq R_i$.

Case (b). Suppose there is no path in $R_i \cup P_1 \cup P_2$ from u to v other than P_1, P_2. Then R_i consists of disjoint components $A, B, C, \ldots$ attached to the vertices of $P_1 \cup P_2$ (see Fig. 10). Let r be contained in A attached to a on, say, P_1. The free cop C_3 moves to a (with C_1, C_2 keeping control of P_1 and P_2), and we are back to case (a) with $u = a$, $R_i = A$ whence we may proceed as described there.

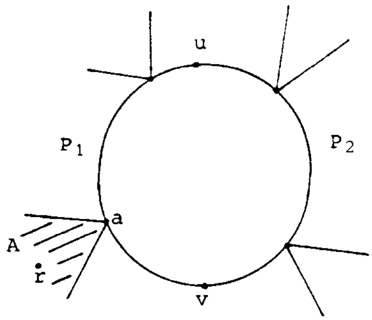

Fig. 10.

Suppose then that there are further u, v-paths in $R_i \cup P_1 \cup P_2$ and let Z be a shortest such path. The following notation is useful: If P is a path and $x, y \in V(P)$, then $P(x, y)$ is the subpath from x to y.

Let w be the first vertex on Z after u which is also on $P_1 \cup P_2$. If $w \in V(P_1)$, then the path $P_3 = Z(u, w) \cup P_1(w, v)$ is, by the minimality of P, also a shortest path which is now disjoint from P_2. Depending on how P_3 partitions R_i we have the two cases of Fig. 11.

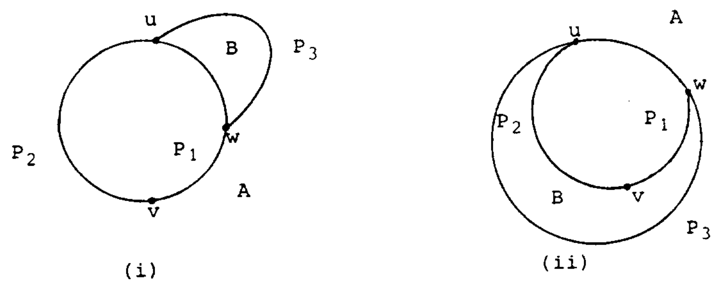

Fig. 11.

Suppose we are in case (i). If r is in A, then the free cop C_3 moves to control P_3, and the pair P_2, P_3; C_2, C_3 gives rise to situation (b) with $R_{i+1} \subsetneq R_i$. (There is at least one vertex on $P_3(u, w)$ which is in R_i but not in R_{i+1}.) If r is in B, then C_1 controls $P_1(u, w)$, C_3 moves to control $P_3(u, w)$ and we are again back to case (b) with $R_{i+1} \subsetneq R_i$. Case (ii) is dealt with by an entirely analogous argument.

Assume, finally, $w \in V(P_2)$. If Z does not intersect P_1 (except in u, v), then we may take $P_3 = Z(u, w) \cup P_2(w, v)$ as another shortest path and repeat the argument just made. Let, otherwise, y be the first intersection of Z with P_1 and x be the preceding intersection of Z with P_2. By the minimality of P_1 and P_2, $P_3 = P_2(u, x) \cup Z(x, y) \cup P_1(y, v)$ is another shortest path (Fig. 12). Now again two situations arise in each of the cases depicted in Fig. 12 depending on whether $r \in A$ or $r \in B$, and the reasoning is as before. □

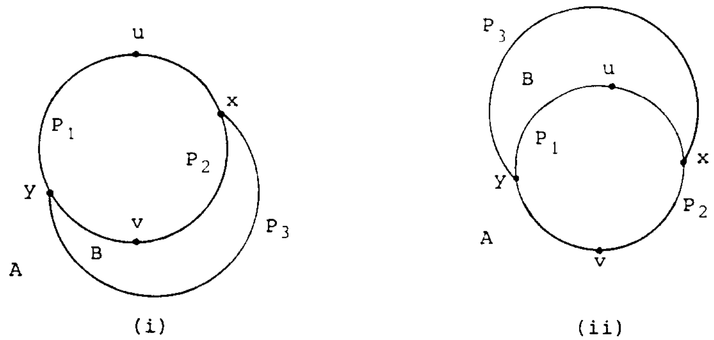

Fig. 12.

The preceding theorem raises the natural question what happens for graphs embedded in the torus or orientable surfaces of higher genus. It seems likely that one has to add two cops when going to the next higher genus. The situation on non-orientable surfaces is probably more involved.

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