Maker Breaker Connectivity

Maker and Breaker play alternatively coloring edge of the complete graph.

Maker chooses 1 uncolored edge and colors it mauve

Breaker chooses 0 uncolored edges and colors them blue

Maker wins if the mauve graph contains a spanning tree, otherwise Breaker wins
Theorem 1 (Gebauer, Szabó)

If \( b \leq b_0 = \frac{n}{\ln n} \left( 1 - \frac{\ln \ln n}{\ln^2 n} - \frac{6}{\ln^2 n} \right) \)

then Maker has a winning strategy.

Proof

We assume that Breaker goes first.

\( B_1, M_1, B_2, M_2, B_3, M_3, \ldots, B_i, M_i \)

\( d(v) = \text{degree of } v \text{ in } B_i \leftrightarrow \text{blue graph} \)

\( C(v) = \text{component of } G_i \leftrightarrow \text{mauve graph containing } v \)
A component \( C(v) \) is dangerous if \( |C_v| \leq 2b \)

\[
\text{danger of } v = \psi(v) = \begin{cases} 
  d_k(v) & \text{if } C_v \text{ is dangerous} \\
  -1 & \text{otherwise}
\end{cases}
\]

Initially, every vertex is active

**Maker's Strategy:** step \( M_t \).

Choose \( u \in C_1 \), an active vertex \( v \) with largest value of \( \psi(v) \).

\( u,v \) is an arbitrary edge \( e \) joining \( C_1(v) \) to \( C_2(v) \). Color \( e \) mauve.

De-activate \( u \).
Observation: Each component of $\mathcal{M}$ has a unique active vertex.

Suppose now that Breaker has a winning strategy.

Let $g$ be first time that Breaker colors all the edges of some cut $(K : \overline{K})$ blue.

$g \leq n-1$ because first $t \leq n-1$ mauve edges form a forest.
Assume $|K| \leq |K|$. 

$|K| < 2b$ else $|K : K'| \geq 2b(n-2b) > bn$. 

Remark 1

$V_i$ is in a dangerous component up to its deactivation.

Remark 2

degree $\geq n - 3b$ just before $B'$s last move.
$v_g$ = arbitrary active vertex at time of $R_t$ last move.

$K$

$v_1, v_2, \ldots, v_{g-1}$ are defined in game.

$I_b = \{ v_{g-b}, \ldots, v_g \}$

For $I \subseteq [n]$ we let

$$
\Psi_{B, t}^I (I) = \frac{1}{|I|} \sum_{v \in I} \psi (v) \text{ \quad just before } R_t
$$
Lemma 1

$1 \leq t \leq g-1$ implies

$$\bar{\Psi}_{M, g-t}(I_t) \geq \bar{\Psi}_{B, g-t+1}(I_{t-1})$$

Proof

$M_g$

$\mathcal{U}_{g-t}, \mathcal{U}_{g-t+1}, \mathcal{U}_{g-t+2}, \ldots, \mathcal{U}_{g}$

dangerous at this time

$B_{g-t+1}$

$\mathcal{U}_{g-t+1}, \mathcal{U}_{g-t+2}, \ldots, \mathcal{U}_{g}$

$I_t$

$I_{t-1}$

$$\bar{\Psi}_{M, g-t}(I_{t-1}) = \bar{\Psi}_{B, g-t+1}(I_{t-1})$$

by choice of $\mathcal{U}_{g-t}$ (dangerous).
Lemma 2

(a) \[ \Psi_{M, g-t}(I_t) - \Psi_{B, g-t}(I_t) \leq \frac{2b}{t+1}. \]

(b) \[ \Psi_{M, g-t}(I_t) - \Psi_{B, g-t}(I_t) \leq \frac{b + t + a(t - 1) - a(t)}{t+1} \]

where \( a(t) = \# \text{edges spanned by } I_t \text{ that } B \text{ took in rounds } 1, 2, \ldots, g-t-1. \)

Proof

(a) \( B \) move does not change \( C(v) \) —
\( B \) move only affects \( d(v) \)

\( \text{Adds at most } b + \text{ed } \leq 2b \)
\( \text{to some of degree in } I_t \)
(b) \[ a(t) + e_d \leq a(t-1) + k \]

- $\#$ blue edges $\leq I_t$
  - chosen in rounds $i, \ldots, g-2$
  - $\geq \#$ edges $\leq I_t$
  - incident to $V_{g-2}$

\[ \square \]

We show $\Psi_{b,1}(I_{g-1}) > 0$ — contradiction.

$\Psi_{b,1}(I_{g-1}) > 0$ — contradiction.

\[ k = \left\lfloor \frac{n}{\ln n} \right\rfloor \]

- $g < k$ and $g \geq k$ treated separately
\[ g < k \]
\[ \overline{\Psi}_{b, i} (I_{g-1}) = \overline{\Psi}_{b, g} (I_0) + \sum_{t=1}^{g-1} \left( \overline{\Psi}_{M, g-t} (I_t) - \overline{\Psi}_{R, g-t+1} (I_t) \right) \]
\[ - \sum_{t=1}^{g-1} \left( \overline{\Psi}_{M, g-t} (I_t) - \overline{\Psi}_{R, g-t} (I_t) \right) \]
\[ \geq n - 3b \]
\[ \geq n - 3b - \sum_{t=1}^{g-1} \frac{b + \delta + a(t-1) - a(t)}{t + 1} \]
\[ \geq n - 3b - b(H_{g-1}) - \frac{a(0)}{2} + \sum_{t=1}^{g-2} \frac{a(t)}{(t+2)(t+3)} + \frac{a(g-1)}{9} \]
\[ \geq n - b(H_g + 2) - 9 \]
\[ \geq n - b(ln k + 3) - k \]
\[ \geq n - \frac{n}{\ln n} (ln n - ln ln n + 3) - \frac{n}{\ln n} > 0. \]
\[
\begin{align*}
\Phi_{g,s_t} (I_{g-1}) &= \Phi_{n,s_t} (I_0) + \sum_{t=1}^{g-1} (\Phi_{M,s_t} (I_t) - \Phi_{\bar{s},s_t} (I_t)) \\
&\quad - \sum_{t=1}^{k-1} (\Phi_{M,s_t} (I_t) - \Phi_{\bar{s},s_t} (I_t)) \\
&\quad - \sum_{t=k}^{g-1} (\Phi_{M,s_t} (I_t) - \Phi_{\bar{s},s_t} (I_t)) \\
&\geq n - 3b - \sum_{t=1}^{k-1} \frac{b + t + a(t-1) - a(t)}{t+1} - \sum_{t=k}^{g-1} \frac{2b}{t+1} \\
&\geq n - 3b - b(H_{k-1}) - (k-1) - \frac{a(0)}{2} \\
&\quad + \sum_{t=1}^{k-2} \frac{a(t)}{(t+2)(t+1)} \cdot \frac{a(k-1)}{k} - 2b(H_g - H_{k-1}) \\
&\geq n - b(2H_g - H_{k} + 2) - k
\end{align*}
\]
\[ \geq n - b(2H_g - H_k + 2) - k \]
\[ \geq n - b(2\ln n - \ln k + 4) - k \]
\[ \geq n - \left( \frac{n}{\ln n} - \frac{n \ln \ln n}{\ln^2 n} - \frac{6n}{\ln^2 n} \right) (\ln n + \ln \ln n + 5) - \frac{n}{\ln n} \]
\[ \geq \frac{n (\ln \ln n)^2}{\ln^2 n} \]
\[ > 0. \]
Theorem 2 (Chvátal, Erdős)

If $b \geq b_1 = \frac{n}{\ln n} (1 + \varepsilon)$ for arbitrary constant $\varepsilon$, then Breaker has a winning strategy.

Proof

Let $k = \lceil \frac{n}{2 \log n} \rceil$. Maker goes first.

After rounds $1 \leq t \leq k$, Breaker can build a unique $C_t$ of size $t$ such that Maker has no edge incident with $C_t$. $t = 1$ is trivial.
Suppose true for $t$.

Breaker more: claim $vw$ plus all edges $wv, wv_j \in C_t$

Maker's next move can only choose one edge incident with one vertex $\notin C_t \cup \{wv, wv_j\}$.

$C = C_k = \{1, 2, \ldots, k\}$ \quad $A_i = \{(i, l): l > k\}$

set $\emptyset$ edges
Claim: Break can claim an $A_i$.

**Proof**

Define $f(1, b) = 0$;

$$f(k, b) = \left\lfloor k \left( f(k-1, b) + b \right) / (k-1) \right\rfloor$$

$$f(k, b) \geq (b-1)k \sum_{i=1}^{k-1} \frac{1}{i} \quad \quad \text{— easy induction on } k.$$
M can choose one element of one $A_i$.
B can choose b elements from $UA_i$.
B wins if he/she claims all elements of some $A_i$.

Claim: B wins if $t < f(k, b)$. $f(2, 6) = 26$

Proof: Induction on $k$. $k = 2$ trivial.

At least M more, B more: $k \rightarrow k-1$

$t \rightarrow t^* \leq t - \lfloor t/k \rfloor - b$

One more

$A_i:\quad f(k, b) - \lfloor t/k \rfloor - b \leq f(k-1, b)$. □
For connectivity game, need only check that

\[ k(n-k) \leq (b-1)k \sum_{i=1}^{k-1} \frac{1}{i} \leq f(k,b) . \]

\[ \uparrow \]

Hedges

C : Z