COMBINATORIAL GAMES

Game 1

Start with n chips. Players A,B alternately take 1,2,3 or 4 chips until there are none left. The winner is the person who takes the last chip:

Example

	A	B	A	B	A	
$n = 10$	3	2	4	1		B wins
$n = 11$	1	2	3	4	1	A wins

What is the optimal strategy for playing this game?

Game 2

Chip placed at point (m, n). Players can move chip to (m', n) or (m, n') where $0 \le m' < m$ and $0 \le n' < n$. The player who makes the last move and puts the chip onto $(0, 0)$ wins.

What is the optimal strategy for this game?

Game 2a Chip placed at point (m, n). Players can move chip to (m', n) or (m, n') or to $(m - a, n - a)$ where $0 \le m' < m$ and $0 \le n' < n$ and $0 \le a \le \min\{m, n\}$. The player who makes the last move and puts the chip onto $(0, 0)$ wins.

What is the optimal strategy for this game?

Game 3

W is a set of words. A and B alternately remove words $w_1, w_2, \ldots,$ from W. The rule is that the first letter of w_{i+1} must be the same as the last letter of w_i. The player who makes the last legal move wins.

Example

$W = \{England, France, Germany, Russia, Bulgaria, \ldots\}$

What is the optimal strategy for this game?

Abstraction

Represent each position of the game by a vertex of a digraph $D = (X, A)$.
(x, y) is an arc of D iff one can move from position x to position y.

We assume that the digraph is finite and that it is **acyclic** i.e. there are no directed cycles.

The game starts with a token on vertex x_0 say, and players alternately move the token to $x_1, x_2, \ldots,$ where $x_{i+1} \in N^+(x_i)$, the set of out-neighbours of x_i. The game ends when the token is on a **sink** i.e. a vertex of out-degree zero. The last player to move is the winner.

Example 1: $V(D) = \{0, 1, \ldots, n\}$ and $(x, y) \in A$ iff $x - y \in \{1, 2, 3, 4\}$.

Example 2: $V(D) = \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$ and $(x, y) \in N^+((x', y')))$ iff $x = x'$ and $y > y'$ or $x > x'$ and $y = y'$.

Example 2a: $V(D) = \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$ and $(x, y) \in N^+((x', y')))$ iff $x = x'$ and $y > y'$ or $x > x'$ and $y = y'$ or $x - x' = y - y' > 0$.

Example 3: $V(D) = \{(W', w) : W' \subseteq W \setminus \{w\}\}$. w is the last word used and W' is the remaining set of unused words. $(X', w') \in N^+((X, w))$ iff $w' \in X$ and w' begins with the last letter of w. Also, there is an arc from $(W, \cdot)$ to $(W \setminus \{w\}, w)$ for all w, corresponding to the games start.

We will first argue that such a game must eventually end.

A **topological numbering** of digraph $D = (X, A)$ is a map $f : X \to [n]$, $n = |X|$ which satisfies $(x, y) \in A$ implies $f(x) < f(y)$.

Theorem

A finite digraph $D = (X, A)$ is acyclic iff it admits at least one topological numbering.

Proof Suppose first that D has a topological numbering. We show that it is acyclic.

Suppose that $C = (x_1, x_2, \ldots, x_k, x_1)$ is a directed cycle. Then $f(x_1) < f(x_2) < \cdots < f(x_k) < f(x_1)$, contradiction.

Suppose now that D is acyclic. We first argue that D has at least one sink.

Thus let $P = (x_1, x_2, \ldots, x_k)$ be a longest simple path in D. We claim that x_k is a sink.

If D contains an arc (x_k, y) then either $y = x_i, 1 \leq i \leq k-1$ and this means that D contains the cycle $(x_i, x_{i+1}, \ldots, x_k, x_i)$, contradiction or $y \notin \{x_1, x_2, \ldots, x_k\}$ and then (P, y) is a longer simple path than P, contradiction.

We can now prove by induction on n that there is at least one topological numbering.

If $n = 1$ and $X = \{x\}$ then $f(x) = 1$ defines a topological numbering.

Now asssume that $n > 1$. Let z be a sink of D and define $f(z) = n$. The digraph $D' = D - z$ is acyclic and by the induction hypothesis it admits a topological numbering, $f : X \setminus \{z\} \to [n-1]$.

The function we have defined on X is a topological numbering. If $(x, y) \in A$ then either $x, y \neq z$ and then $f(x) < f(y)$ by our assumption on f, or $y = z$ and then $f(x) < n = f(z)$ ($x \neq z$ because z is a sink). $\square$

The fact that D has a topological numbering implies that the game must end. Each move increases the f value of the current position by at least one and so after at most n moves a sink must be reached.

The positions of a game are partitioned into 2 sets:

- P-positions: The next player cannot win. The previous player can win regardless of the current player's strategy.
- N-positions: The next player has a strategy for winning the game.

Thus an N-position is a winning position for the next player and a P-position is a losing position for the next player.

The main problem is to determine N and P and what the strategy is for winning from an N-position.

Let the vertices of D be $x_1, x_2, \ldots, x_n$, in topological order.

Labelling procedure

1. $i \leftarrow n$, Label x_n with P. $N \leftarrow \emptyset$, $P \leftarrow \emptyset$.
2. $i \leftarrow i - 1$. If $i = 0$ STOP.
3. Label x_i with N, if $N^+(x_i) \cap P \neq \emptyset$.
4. Label x_i with P, if $N^+(x_i) \subseteq N$.
5. goto 2.

The partition N, P satisfies

$$x \in N \textit{ iff } N^+(x) \cap P \neq \emptyset$$

To play from $x \in N$, move to $y \in N^+(x) \cap P$.

In Game 1, $P = \{5k : \ k \geq 0\}$.

In Game 2, $P = \{(x, x) : \ x \geq 0\}$.

Lemma

The partition into N, P satisfying $x \in N$ iff $N^+(x) \cap P \neq \emptyset$ is unique.

Proof If there were two partitions $N_i, P_i,\ i = 1, 2$, let x_i be the vertex of highest topological number which is not in $(N_1 \cap N_2) \cup (P_1 \cap P_2)$. Suppose that $x_i \in N_1 \setminus N_2$.

But then $x_i \in N_1$ implies $N^+(x_i) \cap P_1 \cap \{x_{i+1}, \ldots, x_n\} \neq \emptyset$ and $x_i \in P_2$ implies $N^+(x_i) \cap P_2 \cap \{x_{i+1}, \ldots, x_n\} = \emptyset$.

But $P_1 \cap \{x_{i+1}, \ldots, x_n\} = P_2 \cap \{x_{i+1}, \ldots, x_n\}$. □

Sums of games

Suppose that we have p games $G_1, G_2, \ldots, G_p$ with digraphs $D_i = (X_i, A_i)$, $i = 1, 2, \ldots, p$.
The sum $G_1 \oplus G_2 \oplus \cdots \oplus G_p$ of these games is played as follows. A position is a vector $(x_1, x_2, \ldots, x_p) \in X = X_1 \times X_2 \times \cdots \times X_p$. To make a move, a player chooses i such that x_i is not a sink of D_i and then replaces x_i by $y \in N_i^+(x_i)$. The game ends when each x_i is a sink of D_i for $i = 1, 2, \ldots, n$.

Knowing the partitions N_i, P_i for game $i = 1, 2, \ldots, p$ does not seem to be enough to determine how to play the sum of the games.

We need more information. This will be provided by the Sprague-Grundy Numbering

Example

Nim

In a one pile game, we start with $a \geq 0$ chips and while there is a positive number x of chips, a move consists of deleting $y \leq x$ chips. In this game the N-positions are the positive integers and the unique P-position is 0.

In general, Nim consists of the sum of n single pile games starting with $a_1, a_2, \ldots, a_n > 0$. A move consists of deleting some chips from a non-empty pile.

Example 2 is Nim with 2 piles.

Sprague-Grundy (SG) Numbering

For $S \subseteq \{0, 1, 2, \ldots, \}$ let

$$mex(S) = \min\{x \geq 0 : \; x \notin S\}.$$

Now given an acyclic digraph $D = X, A$ with topological ordering $x_1, x_2, \ldots, x_n$ define g iteratively by

1. $i \leftarrow n$, $g(x_n) = 0$.
2. $i \leftarrow i - 1$. If $i = 0$ STOP.
3. $g(x_i) = mex(\{g(x) : \; x \in N^+(x_i)\})$.
4. goto 2.

Lemma

$$x \in P \leftrightarrow g(x) = 0.$$

Proof Because

$$x \in N \textit{ iff } N^+(x) \cap P \neq \emptyset$$

all we have to show is that

$$g(x) > 0 \textit{ iff } \exists y \in N^+(y) \textit{ such that } g(y) = 0.$$

But this is immediate from $g(x) = mex(\{g(y) : \ y \in N^+(x)\})$ $\square$

Another one pile subtraction game.

- A player can remove any even number of chips, but not the whole pile.
- A player can remove the whole pile if it is odd.

The terminal positions are 0 or 2.

Lemma

$g(0) = 0$, $g(2k) = k - 1$ *and* $g(2k - 1) = k$ *for* $k \geq 1$.

Proof 0,2 are terminal postions and so $g(0) = g(2) = 0$. $g(1) = 1$ because the only position one can move to from 1 is 0. We prove the remainder by induction on k.

Assume that $k > 1$.

$$\begin{aligned} g(2k) &= mex\{g(2k-2), g(2k-4), \ldots, g(2)\} \\ &= mex\{k-2, k-3, \ldots, 0\} \\ &= k-1. \\ g(2k-1) &= mex\{g(2k-3), g(2k-5), \ldots, g(1), g(0)\} \\ &= mex\{k-1, k-2, \ldots, 0\} \\ &= k. \end{aligned}$$

□

We now show how to compute the *SG* numbering for a sum of games.

For binary integers $a = a_m a_{m-1} \cdots a_1 a_0$ and $b = b_m b_{m-1} \cdots b_1 b_0$ we define $a \oplus b = c_m c_{m-1} \cdots c_1 c_0$ by

$$c_i = \begin{cases} 1 & \text{if } a_i \neq b_i \\ 0 & \text{if } a_i = b_i \end{cases}$$

for $i = 1, 2, \ldots, m$.

So $11 \oplus 5 = 14$.

Theorem

If g_i is the SG function for game G_i, $i = 1, 2, \ldots, p$ then the SG function g for the sum of the games $G = G_1 \oplus G_2 \oplus \cdots \oplus G_p$ is defined by

$$g(x) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_p(x_p)$$

where $x = (x_1, x_2, \ldots, x_p)$.

For example if in a game of Nim, the pile sizes are $x_1, x_2, \ldots, x_p$ then the SG value of the position is

$$x_1 \oplus x_2 \oplus \cdots \oplus x_p$$

Proof It is enough to show this for $p = 2$ and then use induction on p.

Write $G = H \oplus G_p$ where $H = G_1 \oplus G_2 \oplus \cdots \oplus G_{p-1}$. Let h be the SG numbering for H. Then, if $y = (x_1, x_2, \ldots, x_{p-1})$,

$$\begin{aligned} g(x) &= h(y) \oplus g_p(x_p) && \textit{assuming theorem for } p = 2 \\ &= g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_{p-1}(x_{p-1}) \oplus g_p(x_p) && \textit{by induction.} \end{aligned}$$

It is enough now to show, for $p = 2$, that

A1 If $x \in X$ and $g(x) = b > a$ then there exists $x' \in N^+(x)$ such that $g(x') = a$.

A2 If $x \in X$ and $g(x) = b$ and $x' \in N^+(x)$ then $g(x') \neq g(x)$.

A3 If $x \in X$ and $g(x) = 0$ and $x' \in N^+(x)$ then $g(x') \neq 0$

A1. Write $d = a \oplus b$. Then

$$a = d \oplus b = d \oplus g_1(x_1) \oplus g_2(x_2). \qquad (1)$$

Now suppose that we can show that either

$$(i)\ d \oplus g_1(x_1) < g_1(x_1) \text{ or } (ii)\ d \oplus g_2(x_2) < g_2(x_2) \text{ or both.} \qquad (2)$$

Assume that (i) holds.

Then since $g_1(x_1) = mex(N_1^+(x_1))$ there must exist $x_1' \in N_1^+(x_1)$ such that $g_1(x_1') = d \oplus g_1(x_1)$.

Then from (1) we have

$$a = g_1(x_1') \oplus g_2(x_2) = g(x_1', x_2).$$

Furthermore, $(x_1', x_2) \in N^+(x)$ and so we will have verified A1.

Let us verify (2).

Suppose that $2^{k-1} \leq d < 2^k$.

Then d has a 1 in position k and no higher.

Since $d_k = a_k \oplus b_k$ and $a < b$ we must have $a_k = 0$ and $b_k = 1$.

So either (i) $g_1(x_1)$ has a 1 in position k or (ii) $g_2(x_2)$ has a 1 in position k. Assume (i).

But then $d \oplus g_1(x_1) < g_1(x_1)$ since d "destroys" the kth bit of $g_1(x_1)$ and does not change any higher bit.

A2. Suppose without loss of generality that $g(x_1', x_2) = g(x_1, x_2)$ where $x_1' \in N^+(x_1)$.

Then $g_1(x_1') \oplus g_2(x_2) = g_1(x_1) \oplus g_2(x_2)$ implies that $g_1(x_1') = g_1(x_1)$, contradition. $\square$

If we apply this theorem to the game of Nim then if the position x consists of piles of x_i chips for $i = 1, 2, \ldots, p$ then $g(x) = x_1 \oplus x_2 \oplus \cdots \oplus x_p$.

In our first example, $g(x) = x \mod 5$ and so for the sum of p such games we have

$$g(x_1, x_2, \ldots, x_p) = (x_1 \mod 5) \oplus (x_2 \mod 5) \oplus \cdots \oplus (x_p \mod 5).$$

A3. Suppose that $g_1(x_1) \oplus g_2(x_2) = 0$ and $g_1(x_1') \oplus g_2(x_2) = 0$ where $x_1' \in N^+(x_1)$.

Then $g_1(x_1) = g_1(x_1')$, contradicting $g_1(x_1) = mex\{g_1(x) : x \in N^+(x_1)\}$.

A more complicated one pile game

Start with n chips. First player can remove up to $n-1$ chips.

In general, if the previous player took x chips, then the next player can take $y \leq x$ chips.

Thus a games position can be represented by (n, x) where n is the current size of the pile and x is the maximum number of chips that can be removed in this round.

Theorem

Suppose that the position is (n, x) where $n = m2^k$ and m is odd. Then,

(a) *This is an N-position if $x \geq 2^k$.*

(b) *This is a P-position if $m = 1$ and $x < n$.*

Proof For a non-negative integer $n = m2^k$, let $ones(n)$ denote the number of ones in the binary expansion of n and let $k = \rho(n)$ determine the position of the right-most one in this expansion.

We claim that the following strategy is a win for the player in a postion described in (a):

Remove $y = 2^k$ chips.

Suppose this player is A.

If $m = 1$ then $x \geq n$ and A wins.

Otherwise, after such a move the position is (n', y) where $\rho(n') > \rho(n)$.

Note first that $ones(n') = ones(n) - 1 > 0$ and $\rho(n') > k$. B cannot remove more than 2^k chips and so B cannot win at this point.

If B moves the position to (n'', x'') then $ones(n'') > ones(n')$ and furthermore, $x'' \geq 2^{\rho(n'')}$, since x'' must have a 1 in position $\rho(n'')$. ($\rho(n'')$ is the least significant bit of x''.)

Thus, by induction, A is in an N-position and wins the game.

To prove (b), note that after the first move, the position satisfies the conditions of (a). $\Box$.

Let us next consider a generalisation of this game.

There are 2 players A and B and A goes first.

We have a non-decreasing function f from $\mathbf{N} \to \mathbf{N}$ where $\mathbf{N} = \{1, 2, \ldots\}$ which satisfies $f(x) \geq x$.

At the first move A takes any number less than h from the pile, where h is the size of the initial pile.

Then on a subsequent move, if a player takes x chips then the next player is constrained to take at most $f(x)$ chips.

Thus the previous analysis was for the game with $f(x) = x$.

There is a set $\mathcal{H} = \{H_1 = 1 < H_2 < \ldots\}$ of initial pile sizes for which the first player will lose, assuming that the second player plays optimally.

Also, if the initial pile size $h \notin \mathcal{H}$ then the first player has a winning strategy. It will turn out that the sequence satisfies the recurrence:

$$H_{j+1} = H_j + H_\ell \text{ where } H_\ell = \min_{i \leq j}\{H_i \mid f(H_i) \geq H_j\}, \qquad \text{for } j \geq 0.$$

If $f(x) = x$ then $H_j = 2^{j-1}$.

We prove this inductively. It is true for $j = 1$.

$$\begin{aligned} H_{j+1} &= 2^{j-1} + \min_{i \le j}\{2^{i-1} : \ 2^{i-1} \ge 2^{j-1}\} \\ &= 2^{j-1} + 2^{j-1} \\ &= 2^j. \end{aligned}$$

If $f(x) = 2x$ then $\mathcal{H} = \{1, 2, 3, 5, 8, \ldots, \} = \{F_1, F_2, \ldots, \}$, the Fibonacci sequence.

We prove this inductively. It is true for $j = 1, 2$.

$$\begin{aligned} H_{j+1} &= F_j + \min_{i \le j}\{F_i : \; 2F_i \ge F_j\} \\ &= F_j + F_{j-1} \\ &= F_{j+1}. \end{aligned}$$

Recall that $F_j = F_{j-1} + F_{j-2}$ and

$$2F_{j-2} < F_{j-1} + F_{j-2} < 2F_{j-1}.$$

The key to the game is the following result.

Theorem

Every positive integer n can be uniquely written as the sum

$$n = H_{j_1} + H_{j_2} + \cdots + H_{j_p}$$

where $f(H_{j_i}) < H_{j_{i+1}}$ for $1 \leq i < p$.

One simple consequence of the uniqueness of the decomposition is that

$$H_k \neq H_{j_1} + H_{j_2} + \cdots + H_{j_p}$$

for all k and sequences $j_1, j_2, \ldots, j_p$ where $f(H_{j_i}) < H_{j_{i+1}}$ for $i = 1, 2, ..., p-1$.

It follows that the integers n can be given unique "binary" representations by representing $n = H_{j_1} + H_{j_2} + \cdots + H_{j_p}$ by the 0-1 string with a 1 in posiitons $j_1, j_2, \ldots, j_p$ and 0 everywhere else.

Let $\rho_H(n) = p$ be the number of 1's in the representation.

We call this the H-representation of n. This then leads to the following

Theorem

Suppose that the start position is $(n, *)$. Then,

(a) *This is an N-position if $n \notin \mathcal{H} = \{H_1, H_2, \ldots, \}$.*

(b) *This is a P-position if $n \in \mathcal{H}$.*

(a) The winning strategy is to delete a number of chips equal to H_{j_1} where j_1 is the index of the rightmost 1 in the H-representation of $n = H_{j_p} + \cdots + H_{j_1}$.

All we have to do is verify that this strategy is possible.

Note first that if A deletes H_{j_1} chips, then B cannot respond by deleting H_{j_2} chips, because $H_{j_2} > f(H_{j_1})$.

B is forced to delete $x \leq f(H_{j_1}) < H_{j_2}$ chips.

If $p = 2$ then $\rho_H(n - H_{j_1} - x) \geq 1 = \rho_H(n - H_{j_1})$.

If $p \geq 3$ and $y = H_{j_2} - x = H_{k_q} + \cdots + H_{k_1}$ then the H-representation of $n - H_{j_1} - x$ is

$$H_{j_p} + \cdots + H_{j_3} + H_{k_q} + \cdots + H_{k_1}.$$

Here we use the fact that $f(H_{k_q}) \leq f(y) \leq f(H_{j_2}) < H_{j_3}$.

And so in both cases $\rho_H(n - H_{j_2} - x) \geq \rho_H(n - H_{j_1})$ it is only A that can reduce ρ_H.

The next thing to check is that if A starts in $(n, *)$ then A can always delete H_{j_1} chips i.e. the positions (m, x) that A will face satisfy $f(x) \geq H_{k_1}$ where $m = H_{k_1} + H_{k_2} + \cdots + H_{k_q}$.

We do this by induction on the number of plays in the game so far.

It is true in the first move and suppose that it is true for (m, x) and that A removes H_{k_1} and B removes y where $y \leq \min\{m - H_{k_1}, f(H_{k_1})\} < H_{k_2}$. Now if $H_{k_2} - y = H_{\ell_r} + H_{\ell_{r-1}} + \cdots + H_{\ell_1}$ then

$$\begin{aligned} m - H_{k_1} - y &= H_{k_q} + \cdots + H_{k_3} + H_{k_2} - y \\ &= H_{k_q} + \cdots + H_{k_3} + H_{\ell_r} + H_{\ell_{r-1}} + \cdots + H_{\ell_1} \end{aligned}$$

and we need to argue that $H_{\ell_1} \leq f(y)$.

But if $f(y) < H_{\ell_1}$ then we have

$$\begin{aligned} H_{k_2} &= y + H_{\ell_1} + H_{\ell_2} + \cdots + H_{\ell_r} \\ &= H_{a_1} + \cdots + H_{a_s} + H_{\ell_1} + H_{\ell_2} + \cdots + H_{\ell_r} \end{aligned}$$

where $f(H_{a_s}) \leq f(y) < H_{\ell_1}$, which gives two distinct decompositons for H_{k_2}, contradiction.

Thus A can remove H_{ℓ_1} in the next round, as required.

(b) Assume that $n = H_k$. After A removes x chips we have

$$H_k - x = H_{j_1} + H_{j_2} + \cdots + H_{j_p}$$

chips left.

All we have to show is that B can now remove H_{j_1} chips i.e. $H_{j_1} \leq f(x)$.

But if this is not the case then we argue as above that $H_k = H_{a_1} + \cdots + H_{a_s} + H_{j_1} + H_{j_2} + \cdots + H_{j_p}$, where $x = H_{a_1} + \cdots + H_{a_s}$ and $f(H_{j_1}) \leq f(x) < H_{j_1}$, which gives two distinct decompositons for H_k, contradiction.

Proof of the existence of a unique decomposition

We prove this by induction on n. If $n = 1$ then $n = H_1$ is the unique decomposition.
Going back to the defining recurrence we see that

$$H_{j+1} = H_j + H_\ell \leq 2H_j.$$

Existence
Assume that any $n < H_k$ can be represented as a sum of distinct H_{j_i}'s with $f(H_{j_i}) < H_{j_{i+1}}$ and suppose that $H_k \leq n < H_{k+1}$. $H_{k+1} \leq 2H_k$ implies that $n - H_k < H_k$.
It follows by induction that

$$n - H_k = H_{j_1} + \cdots + H_{j_p},$$

where $f(H_{j_i}) < H_{j_{i+1}}$ for $i = 1, 2, ..., p - 1$.

We now need only show that $f(H_{j_p}) < H_k$.

Assume to the contrary that $f(H_{j_p}) \geq H_k$.

Then for some $m \leq j_p$ we have

$$H_{k+1} = H_k + H_m \leq H_k + H_{j_p} \leq n,$$

contradicting the choice of n.

Uniqueness

We will first prove by induction on p that if $f(H_{j_i}) < H_{j_{i+1}}$ for $1 \leq i < p$ then

$$H_{j_1} + H_{j_2} + \cdots + H_{j_p} < H_{j_p+1}. \tag{3}$$

If $p = 2$ then we are saying that if $f(H_{j_1}) < H_{j_2}$ then $H_{j_1} + H_{j_2} < H_{j_2+1}$. But this follows directly from $H_{j_2+1} = H_{j_2} + H_m$ where $f(H_m) \geq H_{j_2}$ i.e. $H_m > H_{j_1}$.

So assume that (3) is true for $p \geq 2$. Now

$$H_{j_{p+1}+1} = H_{j_{p+1}} + H_m \text{ and } f(H_{j_p}) < H_{j_{p+1}}$$

implies that $m \geq j_p + 1$.

Thus

$$\begin{aligned} H_{j_{p+1}+1} &\geq H_{j_{p+1}} + H_{j_p+1} \\ &> H_{j_{p+1}} + H_{j_p} + H_{j_{p-1}} + \cdots + H_{j_1} \end{aligned}$$

after applying induction to get the second inequality.

This completes the induction for (3).

Now assume by induction on k that $n < H_k$ has a unique decomposition. This is true for $k = 2$ and so now assume that $k \geq 2$ and $H_k \leq n < H_{k+1}$. Consider a decomposition

$$n = H_{j_1} + H_{j_2} + \cdots + H_{j_p}.$$

It follows from (3) that $j_p = k$. Indeed, $j_p \leq k$ since $n < H_{k+1}$ and if $j_p < k$ then $H_{j_1} + H_{j_2} + \cdots + H_{j_p} < H_{j_p+1} \leq H_k$, contradicting our choice of n. So H_k appears in every decomposition of n.

Now $H_{k+1} \leq 2H_k$ and $n < H_{k+1}$ implies $n - H_k < H_k$ and so, by induction, $n - H_k$ has a unique decompositon. But then if n had two distinct decompositions, H_k would appear in each, implying that $n - H_k$ also had two distinct decompositions, contradiction.

Note that although we know the optimal strategy for this game, we do not know the Sprague-grundy numbers and so we do not immediately get a solution to multi-pile versions.

This is Game 2a.

Theorem

The set of P-positions is $\mathcal{A} = ((a_i, b_i),\ i = 0, 1, 2, \ldots)$ where $a_i < b_i$, $i \neq 0$ can be generated as follows: $a_0 = b_0 = 0$ and

- *a_i is the smallest integer not appearing in $a_0, b_0, \ldots, a_{i-1}, b_{i-1}$*
- *$b_i = a_i + i$.*

The sequence $\mathcal{A}$ starts

0	0	1	2	3	5
4	7	6	10	8	13
9	15	11	18	12	20
14	23	16	26	17	28
19	31	21	34	22	36
24	39	25	41	27	44

Proof We first prove that each positive integer appears exactly once either as a_i or b_i.

We cannot have $a_i = a_j$ for $i < j$ because a_j is the smallest integer that has not previously appeared. Similarly, we cannot have $a_i < a_{i-1}$, else a_{i-1} was too large.

Since $b_i = a_i + i$ we see that both of the sequences $a_0, a_1, \ldots,$ and $b_0, b_1, \ldots,$ are monotone increasing.

Suppose then that $x = a_i = b_j$. Since $a_i < b_i < b_j$ for $i < j$, we must have $i > j$ here. But then a_i is not an integer that has not appeared before.

Thus each positive integer appears exactly once either as a_i or b_i.

Now suppose that $(a_i, b_i) \in \mathcal{A}$. We consider the possible positions we can move to and check that we cannot move to $\mathcal{A}$:

1. $(a_i - x, b_i) = (a_j, b_j)$ where $x > 0$.
   We must have $j < i$ and $b_j = b_i$. Not possible.
2. $(a_i, b_i - x) = (a_j, b_j)$ where $x > 0$.
   We must have $j < i$ and $a_j = a_i$. Not possible.
3. $(a_i - x, b_i - x) = (a_j, b_j)$ where $x > 0$.
   We must have $j < i$ and $i = b_i - a_i = b_j - a_j = j$. Not possible.

Now suppose that $(c, d) \notin \mathcal{A}$, c, d. We see that we can move to a pair in $\mathcal{A}$.

1. $c = a_i$ and $d > b_i$.
   We can move to (a_i, b_i) by removing $d - b_i$ from the d pile.
2. $c = a_i$ and $d < b_i$.
   Let $j = d - c$. We can move to (a_j, b_j) by deleting $c - a_j = d - b_j$ from each pile.
3. $d = b_i$ and $c > a_i$.
   We can move to (a_i, b_i) by removing $c - a_i$ from the c pile.
4. $d = b_i$ and $c < a_i$ and we are not in Case 1 (with i replaced by i').
   Thus, $c = b_j$ for some $j < i$. We can move to (a_j, b_j) by removing $d - a_j$ from the d pile.

We have therefore verified that the sequence $\mathcal{A}$ does indeed define the set of P positions.

We can give the following description of the sequence $\mathcal{A}$.

Theorem

$$a_k = \lfloor \tfrac{k}{2}(1+\sqrt{5}) \rfloor \text{ and } b_k = \lfloor \tfrac{k}{2}(3+\sqrt{5}) \rfloor$$

for $k = 0, 1, 2, \ldots$.

Proof It will be enough to show

Each non-negative integer appears exactly once in the sequence $(x_k, y_k) = \left(\lfloor \frac{k}{2}(1+\sqrt{5}) \rfloor, \lfloor \frac{k}{2}(3+\sqrt{5}) \rfloor\right)$ $(*)$.

Given (*) we assume inductively that $(a_i, b_i) = (x_i, y_i)$ for $0 \le i \le k$. This is true for $k = 0$.

Using (*) we see that a_{k+1} appears in some pair x_j, y_j. We must have $j > k$ else a_{k+1} will appear in $a_0, \ldots, b_k$.

Now x_{k+1} is the smallest integer that that does not appear in $(x_0, \ldots, y_k) = (a_0, \ldots, b_k)$ and so $x_{k+1} = a_{k+1}$ and then $y_{k+1} = x_{k+1} + k = b_{k+1}$, completing the induction.

Proof of (*)

Fix an integer n and write

$$\alpha = \frac{1}{2}p(1+\sqrt{5}) - n \quad (4)$$

$$\beta = \frac{1}{2}q(3+\sqrt{5}) - n \quad (5)$$

where p, q are integers and

$$0 < \alpha < \frac{1}{2}p(1+\sqrt{5}) \quad (6)$$

$$0 < \beta < \frac{1}{2}q(3+\sqrt{5}) \quad (7)$$

Multiply (4) by $\frac{1}{2}(-1+\sqrt{5})$ and (5) by $\frac{1}{2}(3-\sqrt{5})$ and add to get

$$\frac{1}{2}\alpha(-1+\sqrt{5})+\frac{1}{2}\beta(3-\sqrt{5}) = p+q-n = \ \mathit{integer}.$$

Multiply (6) by $\frac{1}{2}(-1+\sqrt{5})$ and (7) by $\frac{1}{2}(3-\sqrt{5})$ and add to get

$$0 < \frac{1}{2}\alpha(-1+\sqrt{5})+\frac{1}{2}\beta(3-\sqrt{5}) < 2.$$

We see therefore that

$$\frac{1}{2}\alpha(-1+\sqrt{5})+\frac{1}{2}\beta(3-\sqrt{5}) = p+q-n = 1. \qquad (8)$$

Although $\alpha=\beta=1$ satisfies (8) this can be rejected by observing that (4) would then imply that $n+1=p(1+\sqrt{5})$.

Thus either (i) $\alpha < 1, \beta > 1$ or (ii) $\alpha > 1, \beta < 1$.

In case (i) we have from (4) that $n = \lfloor p(1+\sqrt{5})\rfloor$, while in case (ii) we have from (5) that $n = \lfloor q(3+\sqrt{5})\rfloor$

This proves that n appears among the x_k, y_k.

We now argue that the x_k, y_k are distinct.

In Case (i) we can that since $\beta > 1$ is as small as possible, $n \neq y_k$ for every k. In Case (ii) we see that $n \neq x_k$ for every k.

So if an n appears twice, then we would have (a) $x_k = x_\ell$ or (b) $y_k = y_\ell$ for some $k > \ell$.
But (a) implies $0 = x_k - x_\ell = \frac{1}{2}(k-\ell)(1+\sqrt{5}) - \eta$ where $|\eta| < 1$, a contradiction. We rule out (b) in the same way.

Geography

Start with a chip sitting on a vertex v of a graph or digraph G. A move consists of moving the chip to a neighbouring vertex.

In edge geography, moving the chip from x to y deletes the edge (x, y). In vertex geography, moving the chip from x to y deletes the vertex x.

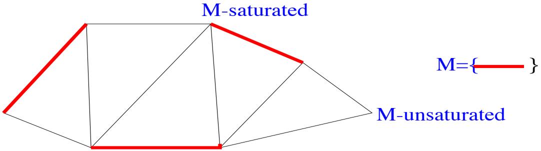
The problem is given a position (G, v), to determine whether this is a P or N position.

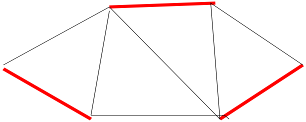
Complexity Both edge and vertex geography are Pspace-hard on digraphs. Edge geography is Pspace-hard on an undirected graph. Only vertex geography on a graph is polynomial time solvable.

Undirected Vertex Geography

We need some simple results from the theory of matchings on graphs.

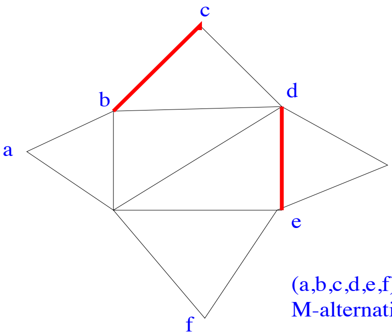
A *matching* M of a graph $G = (V, E)$ is a set of edges, no two of which are incident to a common vertex.





Perfect Matching

M-alternating path



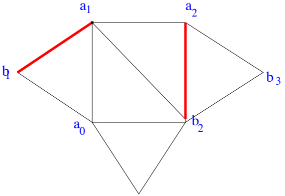
An *M*-alternating path joining 2 *M*-unsaturated vertices is called an *M*-augmenting path.

is a *maximum* matching of G if no matching M' has more edges.

Theorem

M is a maximum matching iff M admits no M-augmenting paths.

Proof Suppose M has an augmenting path
$P = (a_0, b_1, a_1, \ldots, a_k, b_{k+1})$ where
$e_i = (a_{i-1}, b_i) \notin M$, $1 \leq i \leq k+1$ and
$f_i = (b_i, a_i) \in M$, $1 \leq i \leq k$.



$$M' = M - \{f_1, f_2, \ldots, f_k\} + \{e_1, e_2, \ldots, e_{k+1}\}.$$

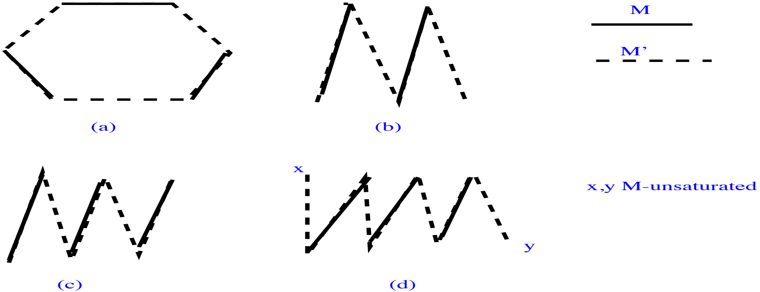
- $|M'| = |M| + 1$.
- M' is a matching

For $x \in V$ let $d_M(x)$ denote the degree of x in matching M, So $d_M(x)$ is 0 or 1.

$$d_{M'}(x) = \begin{cases} d_M(x) & x \notin \{a_0, b_1, \ldots, b_{k+1}\} \\ d_M(x) & x \in \{b_1, \ldots, a_k\} \\ d_M(x) + 1 & x \in \{a_0, b_{k+1}\} \end{cases}$$

So if M has an augmenting path it is not maximum.

Suppose M is not a maximum matching and $|M'| > |M|$. Consider $H = G[M \nabla M']$ where $M \nabla M' = (M \setminus M') \cup (M' \setminus M)$ is the set of edges in *exactly* one of M, M'.
Maximum degree of H is 2 – ≤ 1 edge from M or M'. So H is a collection of vertex disjoint alternating paths and cycles.



$|M'| > |M|$ implies that there is at least one path of type (d). Such a path is M-augmenting $\square$

Theorem

(G, v) is an N-position in UVG iff every maximum matching of G covers v.

Proof (i) Suppose that M is a maximum matching of G which covers v. Player 1's strategy is now: Move along the M-edge that contains the current vertex.

If Player 1 were to lose, then there would exist a sequence of edges $e_1, f_1, \ldots, e_k, f_k$ such that $v \in e_1$, $e_1, e_2, \ldots, e_k \in M$, $f_1, f_2, \ldots, f_k \notin M$ and $f_k = (x, y)$ where y is the current vertex for Player 1 and y is not covered by M.

But then if $A = \{e_1, e_2, \ldots, e_k\}$ and $B = \{f_1, f_2, \ldots, f_k\}$ then $(M \setminus A) \cup B$ is a maximum matching (same size as M) which does not cover v, contradiction.

(ii) Suppose now that there is some maximum matching M which does not cover v.

If (v, w) is Player 1's move,then w must be covered by M, else M is not a maximum matching.

Player 2's strategy is now: Move along the M-edge that contains the current vertex. If Player 2 were to lose then there exists $e_1 = (v, w), f_1, \ldots, e_k, f_k, e_{k+1} = (x, y)$ where y is the current vertex for Player 2 and y is not covered by M.

But then we have defined an augmenting path from v to y and so M is not a maximum matching, contradiction. $\square$

Note that we can determine whether or not v is covered by all maximum matchings as follows: Find the size σ of the maximum matching G.

This can be done in $O(n^3)$ time on an n-vertex graph. Then find the size σ' of a maximum matching in $G - v$. Then v is covered by all maximum matchings of G iff $\sigma \neq \sigma'$.