Notes based on
Richman games
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Rich Man Games

Digraph \( D = (V, A) \) \( |V| < \infty \)

2 distinguished vertices \( r, b \)

2 players \( \text{RED}, \text{BLUE} \)

Token sits at vertex \( v \in A \).

Red has \( X_r \) units of money

Blue has \( X_B \) units of money.

For every vertex \( v \) there is a path from \( v \) to \( r \) or \( b \) or both.
More: RED bids $c \leq X_R$

BLUE bids $y \leq X_B$

If $c > y$ then RED move token

$X_R \leftarrow X_R - x$

$X_B \rightarrow X_B + x$

If $c < y$ then BLUE move token.

$X_R \leftarrow X_R + y$

$X_B \leftarrow X_B - y$

If $c = y$ then choice of move is random
RED is trying to move token to r
BLUE is trying to move token to b.

\[ \text{Thm} \]
Assume \( X_B + X_R = 1 \)

\( \exists R : V \rightarrow [0, 1] \)
such that \( \forall X_B > R(V) \), then
BLUE wins in finite number of
more.
For a vertex \( v \) let \( S(v) \) be the set of all successors of \( v \).

For a function \( f : V \to \mathbb{R} \)

define

\[
    f^+(v) = \max_{\omega \in S(v)} f(\omega)
\]

and

\[
    f^-(v) = \min_{\omega \in S(v)} f(\omega)
\]

if
\( R : V \to [0,1] \) is a RICHMAN function

- \( R(b) = 0 \)
- \( R(\emptyset) = 1 \)
- \( R(v) = \frac{R^+(v) + R^-(v)}{\forall v \in V} \)

Lemma

D has a RICHMAN function

Proof

Let \( R(b, t) = 0 \), \( t = 0, 1, 2, \ldots \)
- \( R(\emptyset, t) = 1 \), \( t = 0, 1, 2, \ldots \)
- \( R(v, 0) = 1 \), \( v \in V \)

\[ R(v, t) = \frac{1}{2} \left( R^+(v, t-1) + R^-(v, t-1) \right) \]

for \( t = 1, 2, \ldots \).
\[ R(v, 1) \leq R(v, 0) \quad \forall v \in V \]

induction

\[ R(v, t+1) \leq R(v, t) \quad \forall v \in V \]

Thus

\[ R(v, t) \rightarrow R(v) \text{ as } t \rightarrow \infty \]

and \( R(v) \) is a PICHMAN Function.

**Proof**

Identify \([f : V \rightarrow [0, 1]^F] \) with \([0, 1]^V \)

\[ \Phi(f) = g \text{ where } g(v) = \frac{1}{2}(f^+(v) + f^-(v)) \]

\( \Phi \) is a continuous map.

Has a fixed point [Brouwer] \( \square \)
Theorem

Let \( R(v,t) \) be as defined in proof 1.

Suppose initially that \( X+Y < 1 \).

If \( Y > R(v,t) \) then Blue wins in \( \delta \) move from \( v \).

Proof

By induction on \( t \).

Trivial for \( t = 0 \).
Assume true for $b-1$.

\[ R(u_{j,b-1}) = R^-(u_{j,b-1}) \]

\[ R(w_{j,b-1}) = R^+(w_{j,b-1}) \]

Blue bids $\Delta = \frac{1}{2} (R(w_{j,b-1}) - R(u_{j,b-1}))$

(1) If $B$ wins bid then he moves to $u$ and has $\Delta = R(u_{j,b}) - \Delta = R(u_{j,b-1})$ and wins in $b-1$ moves.

(1) If $R$ wins bid and moves to $u$ then $B$ now has more than

\[ R(u_{j,b}) + \Delta = R(w_{j,b-1}) \geq R(x_{j,b-1}) \]

and he will again win.
One can define

\[ R'(b, t) = 0, \quad b = 0, 1, 2, \ldots \]
\[ R'(r, t) = 1, \quad t = 0, 1, 2, \ldots \]
\[ R'(v, 0) = 0, \quad v \in V \]

and

\[ R'(v, 2) = \frac{1}{2} (R'^+ (v, t-1) + R'^-(v, t-1)) \]

\[ R'(v, 2) \xrightarrow{\text{Richman}} R'(v) \]

minimum amount of money that B needs to prevent R winning in t steps.
**Thm**

*Richman* function is unique 
($R' = R$)

**Proof**

Let $A(v) = \{ w : w \text{ readable from } v \text{ by edges of steepest descent} \}$

Edge of steepest descent
Claim

\[ R(z) < 1 \implies b \in A(z) \]

Proof

Choose \( w \in A(z) \) s.t.

\[ R(w) = \min \{ R(u) : u \in A(z) \} \]

Assume \( w \neq b \)

\[ R(w) = R^{-1}(w) \]

\[ R(u) = R(v) \]

\[ R(w) = \frac{R(w) + R(w)}{2} \]

\[ R(w) = R(w) \]
So, b can be reached from v, proving the claim.

All have the same R value.

\[ R(v) = 1 \implies R(v) \]

r cannot be reached from v.
Suppose now that $R_1, R_2$ are both Rice-Man functions.

Choose $v$ s.t. $R_1 - R_2$ is maximized at $v$.

$$M = R_1(v) - R_2(v).$$

\[ R_1(w) = R_1^+(v) \]
\[ R_1(w) = R_1^-(v) \]
\[ R_2(u_1) = R_2^+(v) \]
\[ R_2(u_2) = R_2^-(v) \]
\[ R_2(w_2) = R_2^+(v) \]
\[ R_1(u_1) - R_2(u_2) \leq R_1(u_2) - R_2(u_2) \leq M \]
\[ R_1(w_1) - R_2(w_2) \leq R_1(w_1) - R_2(w_1) \leq M \]

\[ \frac{1}{2} (R_1(u_1) + R_1(w_1)) - \frac{1}{2} (R_2(u_2) + R_2(w_2)) \leq 2M \]

So we have \( M \) in above

\[ u \in A_2(w) \]

\[ R_1(w) - R_2(w) = M \]
If $R_2(v) < 1$ then $b \in A_2(v)$
and so $M \leq 0$.

Similar argument shows $M \geq 0$

What if $R_2(v) = 1$?
Clearly $M \leq 0$ now.
Incomplete Knowledge

Define Blue's safety ratio at $u$

$\rho(u) = \frac{\lambda}{R(u)}$

$\lambda = \frac{x_B}{x_B + x_R}$

Thm

If $\rho(u) > 1$ then B can win with probability 1 without knowing R’s cash.
Blue strategy: act as if $\rho(w) = 1$.

Blue knows $X_B$.

Act as if

$$\frac{X_B}{X_R + X_B} = R(w)$$

i.e.

$$X_R = \left(\frac{1}{R(w)} - 1\right) X_B$$

and bid

$$\Delta = \frac{X_B}{R(w)} \left( R(w) - R^*(w) \right)$$
Case 1: Blue wins bid.

New safety ratio:

\[
\frac{X_B - \Delta}{X_B + X_R} = \frac{X_B}{X_B + X_R} = R^-(\nu)
\]

i.e. no change!

Case 2: Red wins bid.

New safety ratio (z \geq \Delta)

\[
\geq \frac{X_B + z}{X_B + X_R} \geq \frac{X_B + \Delta}{X_R + R} = \frac{R^+(\nu)}{R^+(\nu)}
\]
Thus safety ratio is non-decreasing.

If BLUE lost i.e. token moved to 1 then safety ratio would be at most 1 — contradiction.

Game ends in finite time with probability 1.

If 0 is any elicit then game must end in finite time.
Computing $R(v)$:

Solve linear program

$$\min \sum_v x^+_v - \sum_v x^-_v$$

s.t.

$$x^+_v \geq x_w$$

$$x^-_v \leq x_w$$

$$x^+_v = \frac{oc^+_v + oc^-_v}{2}$$

$$0 \leq x^-_v \leq 1$$

$$x^+_0 = 0 \quad x^+_{l_p} = 1$$