Notes based on

Richman games

by

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Rich Man Games

Note Title 4/6/2005

Digraph $D = (V, A)$ $|V| < \infty$

2 distinguished vertices r, b

2 players RED, BLUE

Token sits at vertex $v \in A$.

Red has X_R units of money

Blue has X_B units of money.

For every vertex v there is a path from v to r or b or both.

Move: RED bids $x \leq X_R$

BLUE bids $y \leq X_B$

If $x > y$ then RED moves token

$X_R \leftarrow X_R - x$

$X_B \rightarrow X_B + x$

If $x < y$ then BLUE moves token.

$X_R \leftarrow X_R + y$

$X_B \leftarrow X_B - y$

If $x = y$ then choice of move is random

RED is trying to move token to r

BLUE is trying to move token to b.

Thm

Assume $X_B + X_R = 1$

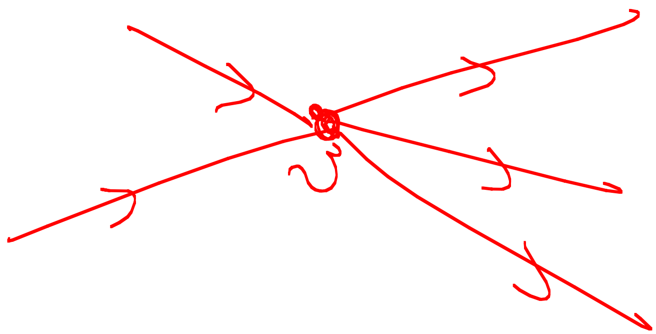
$\exists\, R : V \longrightarrow [0, 1]$

such that: if $X_B > R(v)$, then

Blue wins in finite number of

moves.

For a vertex v let $S(v)$ be successors of v



$S(v)$

For a function $f : V \to \mathbb{R}$

define
$$f^+(v) = \max_{w \in S(v)} f(w)$$
$$f^-(v) = \min_{w \in S(v)} f(w)$$

if

$R : V \to [0,1]$ is a RICHMAN function

(i) $R(b) = 0$, (ii) $R(r) = 1$, (iii) $R(v) = \dfrac{R^+(v) + R^-(v)}{2}$ $\forall v \in V$

Lemma

D has a RICHMAN function

Proof

Let $R(b,t) = 0, \quad t = 0, 1, 2, \ldots$

$R(r,t) = 1, \quad t = 0, 1, 2, \ldots$

$R(v,0) = 1 \quad v \in V$

$$R(v,t) = \frac{1}{2}\left(R^+(v,t-1) + R^-(v,t-1)\right)$$

for $t = 1, 2, \ldots$

$$R(v,1) \le R(v,0) \quad \forall v \in V$$

induction

$$R(v,t+1) \le R(v,t) \quad \forall v \in V$$

Thus

$$R(v,t) \to R(v) \quad \text{as } t \to \infty$$

and $R(v)$ is a RICHMAN Function.

Proof 2

Identify $\{f : V \to [0,1]\}$ with $[0,1]^V$

$$\Phi(f) = g \quad \text{where} \quad g(v) = \tfrac{1}{2}\left(f^+(v) + f^-(v)\right)$$

Φ is a continuous map.

Has a fixed point [Brouwer] $\square$

Theorem

Let $R(v,t)$ be as defined in proof 1.

Suppose initially that $X+Y=1$

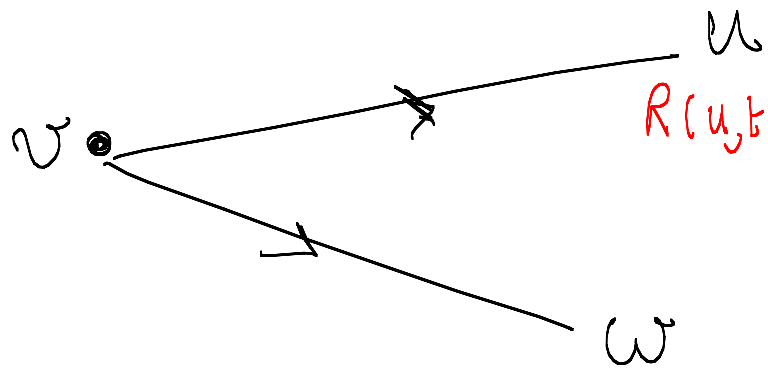
If $Y > R(v,t)$ then Blue wins in $\leq t$ moves from v.

Proof

By induction on t.

Trivial for $t=0$.

Assume true for $t-1$.



$R(u,t-1) = R^{-}(v,t-1)$

$R(w,t-1) = R^{+}(v,t-1)$

Blue bids $\Delta = \frac{1}{2}\left(R(w,t-1) - R(u,t-1)\right)$

(I) If B wins bid then he moves to u and has $\geq R(v,t) - \Delta = R(u,t-1)$ and wins in $t-1$ moves.

(II) If R wins bid and moves to x then B now has more than

$$R(v,t) + \Delta = R(w,t-1) \geq R(x,t-1)$$

and he will again win. $\square$

One can define

$$R'(b,t) = 0, \quad t = 0,1,2,\ldots$$

$$R'(r,t) = 1, \quad t = 0,1,2,\ldots$$

$$R'(v,0) = 0 \quad v \in V$$

and

$$R'(v,t) = \frac{1}{2}\left(R'^{+}(v,t-1) + R'^{-}(v,t-1) \right)$$

$R'(v,t) \nearrow R'(v) \leftarrow$ RICHMAN

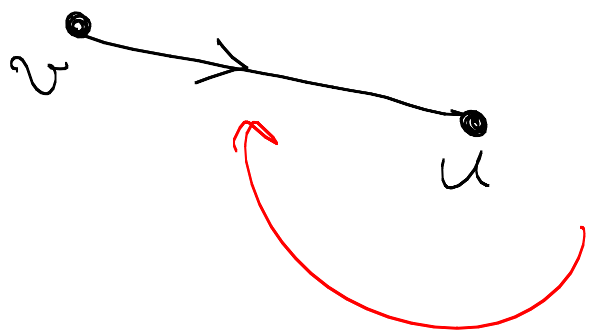
minimum amount of money that B needs to prevent R winning in t steps.

Thm

RICHMAN function is unique

$(R' = R)$

Proof



$R(u) = R^-(v)$

Edge of steepest descent.

Let $A(v) = \{ w : w \text{ reachable from } v \text{ by edges of steepest descent} \}$

Claim

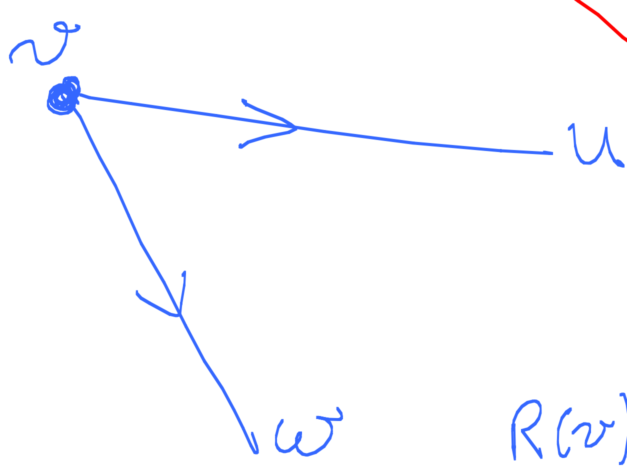
$$R(z) < 1 \implies b \in A(z)$$

Proof

Choose $v \in A(z)$ s.t.

$$R(v) = \min\{R(u) : u \in A(z)\}$$

Assume $v \neq b$



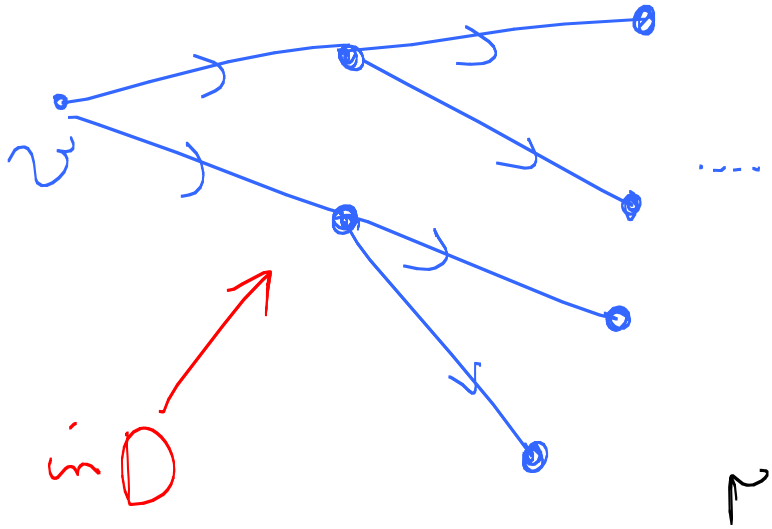
$R(u) = R^{-}(v)$

$\Downarrow$

$R(u) = R(v)$

$$R(v) \geq \frac{R(w) + R(u)}{2}$$

$$R(w) = R(v)$$



All have same R value.

r cannot be reached from v

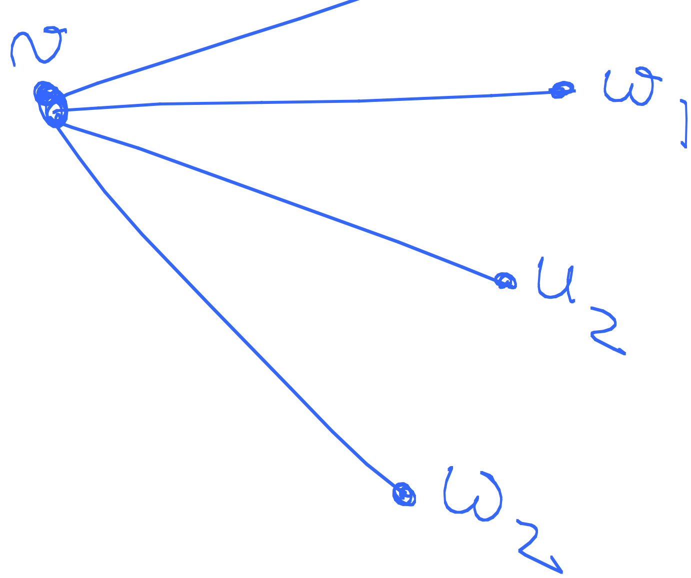
$(R(r) = 1 > R(v))$

So, b can be reached from v, proving claim.

Suppose now that R_1, R_2 are both RICHMAN functions.

Choose v s.t. $R_1 - R_2$ is maximised at v.

$$M = R_1(v) - R_2(v).$$



$$R_1(u_1) = R_1^-(v)$$

$$R_1(w_1) = R_1^+(v)$$

$$R_2(u_2) = R_2^-(v)$$

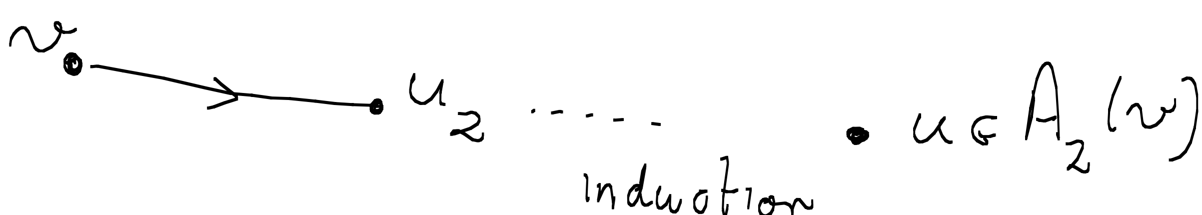
$$R_2(w_2) = R_2^+(v)$$

$$R_1(u_1) - R_2(u_2) \leq R_1(u_2) - R_2(u_2) \leq M$$

$$R_1(w_1) - R_2(w_2) \leq R_1(w_1) - R_2(w_1) \leq M$$

$$\underbrace{\big(R_1(u_1) + R_1(w_1)\big) - \big(R_2(u_2) + R_2(w_2)\big)}_{2\left(R_1(v) - R_2(v)\right)} \leq 2M$$

So we have $=$ in above



$$R_1(u) - R_2(u) = M$$

If $R_2(v) < 1$ then $b \in A_2(v)$ and so $M \leq 0$.

Similar argument shows $M \geq 0$

What if $R_2(v) = 1$?

Clearly $M \lneq 0$ now.

$\square$

Incomplete Knowledge

Define Blue's safety ratio at v

$$\rho(v) = \frac{\lambda}{R(v)} \qquad \lambda = \frac{X_B}{X_R + X_B}$$

Thm

If $\rho(v) > 1$ then B can win with probability 1 without knowing R's cash.

Blue strategy: act as if $\rho(v) = 1$.

Blue knows X_B.

Act as if
$$\frac{X_B}{X_R + X_B} = R(v)$$

i.e.
$$X_R = \left(\frac{1}{R(v)} - 1\right) X_B$$

and bid
$$\Delta = \frac{X_B}{R(v)}\left(R(v) - R^-(v)\right)$$

Case 1: Blue wins bid.

New safety ratio:

$$\frac{\dfrac{X_B - \Delta}{X_B + X_R}}{R^-(v)} = \frac{\dfrac{X_B}{X_B + X_R}}{R(v)}$$

i.e. no change!

Case 2: Red wins bid.

New safety ratio $(z \geq \Delta)$

$$\geq \frac{\dfrac{X_B + z}{X_B + X_R}}{R^+(v)} \geq \frac{\dfrac{X_B + \Delta}{X_B + R}}{R^+(v)} =$$

Thus safety ratio is non-decreasing.

If BLUE lost i.e. token moved to r then safety ratio would be at most 1 — contradiction.

Game ends in finite time with probability 1.

$\square$

If D is acyclic then game <u>must</u> end in finite time.

Computing $R(v)$:

Solve linear program

minimize $$\sum_v x_v^+ - \sum_v x_v^-$$

s.t.

$$x_v^+ \geq x_w \qquad v \to w$$

$$x_v^- \leq x_w$$

$$x_v = \frac{x_v^+ + x_v^-}{2}$$

$$0 \leq x_v \leq 1$$

$$x_b = 0 \qquad x_r = 1$$