15-859: Mathematical Games (CMU Spring 2011, Frieze and Sleator)

Homework 2 (due Thursday, February 17)

Directions: *You may work in small groups on these problems, but write your own solutions.*

1. LOL (40 points)

"LOL" is a two player game, where the two players alternate moves. The position in this game is a string of characters from the set { `L`, `O`, `-` }. A move consists of the replacement of a `-` by an `L` or by an `O`. If a player creates a region of three consecutive characters that are "`LOL`" then that player immediately wins. If no moves are possible, then the game is a draw.

Clearly, starting from any position P of the game, there are only three possibilities for the outcome: The first player wins, the second player wins, or it's a draw. Let us assume in this problem that both players play optimally, that is, the player with the winning strategy forces a win in the fewest possible moves, the player who will lose forces the game to go on as long as possible. (In case of a draw the length of the game is equal to the number of `-` characters in the position.) We're interested in computing for a given position P, a number $N(P)$, which is how the number of moves the game will last.

(a) [10 Points] Suppose we start with a string of n "`-`" characters. Prove that if n is odd the second player cannot win, and if n is even, the first player cannot win.

(b) [10 Points] Here we extend the result in part (a). Consider an arbitrary position P which does not admit an instant winning move. Prove that if there are an odd number of "`-`"s then the second player cannot win, and if there are an even number of "`-`"s then the first player cannot win. (Hint: it may be useful to change the game slightly. Just as in chess it's illegal to move your king into check, we can modify the game and make it forbidden to create a position from which the opponent can make an `LOL`. This will not change the outcome of the game.)

(c) [10 Points] Let **LEFT** be the player (identified in part (b)) who moves when there are an odd number of "`-`"s left (i.e. the player who might win). Let **RIGHT** be the other player.

Let's define a game called "LOLY", which as a slightly modified version of LOL. We allow **LEFT** to make a move that replaces `L--L` by `LLLL`. We don't allow **R** to make this kind of move. We also don't allow (as explained above) any move which would let the other player complete an `LOL`.

LOLY is a partizan game for which we can compute the combinatorial game value. For example, the value of `-` is *, the value of `L--L` is 1. Compute the values of the following positions:

`---`   `L---`   `L---L`   `-----`   `L----L`

(d) [10 Points] Suppose P is this position: `L-------------------L`, (that's 19 `-` characters) and suppose that you know that its combinatorial game value is $\{\{\{4|3\}|\{2|2\}\}|1\}$. What is the value of $N(P)$, and why?

Explain how this connection to combinatorial game theory might help make computing the values of $N(P)$ more efficient.

2. Cops and Robbers on a Grid (20 points)

The $n \times n$ grid is a graph where each vertex is an ordered pair of integers (i, j) where $1 \leq i, j \leq n$. A point (a, b) neighbors a point (c, d) if $(a - c)^2 + (b - d)^2 = 1$.

(a) [10 Points] We know from the planar cops and robbers theorem that the cop number of a 2-d grid is at most 3. Give a winning procedure for 2 cops on a 2-d grid, or prove that there is no such strategy.

(b) [10 Points] Try to generalize your result from part (a) to a d-dimensional grid.

3. Cops and Robbers on a Hypercube (30 points)

The d-dimensional hypercube is a graph of 2^d vertex, where each vertex is a bit vector of d bits, and two vertices are neighbors if they differ in exactly one bit.

The cop number of a 3-d hypercube is 2. Give lower and upper bounds (the best you can come up with) for the cop number of the d-dimensional hypercube.