15-859: Mathematical Games (CMU Spring 2011, Frieze and Sleator)

Homework 1 (due Thursday, February 3)

Directions: *You may work in small groups on these problems, but write your own solutions.*

1. 2-D Coin Moving Game (10 points)

Each square of an $n \times n$ board has a pile of zero or more coins on it. A move consists of picking some coins (at least one) on a square and moving them up one square, or to the left one square. The game is over when all the coins are on the upper left square. Give a formula for the Grundy number of a position in this game.

2. 1-D Coin Sliding Game (10 points)

n coins are arranged on a $1 \times m$ grid, but this time coins may not be stacked on top of one another. Adam and Bonnie take turns sliding a single coin any number of spaces to the left (without skipping over other coins, or landing on another coin). When the coins are positioned on the left-most n squares, the last player who moved wins. Give a formula for the Grundy number of a position in this game.

3. Coin Flipping Game (10 points)

n coins are arranged in a row with either heads or tails showing. Two boys take turns flipping the coins over. During each player's turn, he must choose a single coin showing heads and flip it to tails. He may then either end his turn immediately, or flip any other coin to the left of the coin he chose and end his turn. When all coins show tails, the player who just moved wins. Give a formula for the Grundy number of a position in this game.

4. Variation on Light's Out (20 points)

Consider an electronic game that consists of an $n \times n$ array of lights. Each bulb is always either on or off. The goal is to take an arbitrary initial pattern of lights, and turn as many off as possible. A subset of n positions of the $n \times n$ square is called a *permutation* if the subset has exactly one chosen element in each row and each column. A move consists of toggling the state of all the lights of a permutation. Prove that if n is odd, it's always possible (using a sequence of permutations) to reduce the number of lighted cells to at most $n - 1$. Can you do better than this? What if n is even?